

DISORDERED CONDUCTORS IN A
RANDOM MATRIX FORMULATION
AND THE CONNECTION TO COMPLEX SYSTEMS

By

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For my mate ...

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Complex physical quantum systems represent a diverse variety of problems in theoretical physics, ranging from mesoscopic physics to quantum chaos. The challenge consists in extracting similar and universal features from apparently different systems.

The energy spectrum is such a common characteristic, revealing ultimately important clues about the nature of these systems. Random matrix theory is an important tool to identify the universal part and also to provide a semiphenomenologic theory for the system-specific behaviour. Standard random matrix theory is very successful in explaining many features of the energy spectra, but is restricted to systems exhibiting chaotic spectra.

It is a natural approach to extend this theory to include systems that show a transition from the highly correlated chaotic towards an uncorrelated integrable spectral behaviour. Of special interest is the metal-insulator transition in the disordered conductor problem where the metallic regime is chaotic while the insulating regime is integrable.

The approach used here defines a generalized random matrix ensemble, based on

numerical studies of microscopic models of the metal-insulator transition. It introduces an internal parameter, which can be considered as the measure of chaos in the spectra or the measure of disorder in the system. The significance of this ensemble is that it is exactly solvable and, therefore, permits on a reliable basis the in-depth study of the corresponding physical properties. The generalized ensemble belongs to a different universality class and is therefore already interesting in its own right.

The general features of this new ensemble are discussed, and a detailed statistical analysis is performed, allowing the comparison with other investigations of either numerical or experimental, available in the literature.

Furthermore, the model is expanded to include effects of an external parameter like an magnetic field. This requires a dynamic formulation of the random matrix approach, which was pioneered by Dyson. The expanded model for the generalized ensemble is compared to the standard case and predictions for physical systems exhibiting transitional statistics are made.

CHAPTER 1

INTRODUCTION

In the area of quantum physics complex systems have remained a challenging and rewarding subject to investigate. The challenge lies in the absence of exact solutions and the ineffectivity of perturbation approaches due to nonlinear resonances. However, random matrix theory has proven to be quite useful for a partial description of the system, mostly in the chaotic region. Until recently a random matrix model, which can also capture the physical aspects of systems whose characteristics lie between regular and chaotic behavior, was not available.

The main purpose of this thesis is to motivate, introduce and investigate a generalized random matrix model, which can describe certain systems in the crossover region between chaotic and regular behavior. Before detailing the main model, it is useful to review the general methods necessary for the investigation of the approach.

1.1 Random Matrix Theory

Introduced first by Wigner in the 1950s [1, 2, 3], random matrix theory (RMT) is a versatile tool for analysing a variety of complex systems. Originally envisioned to be applied to the energy eigenvalue spectrum of nuclei, the usefulness of RMT has expanded into solid state physics and into nonlinear and chaotic systems. The pioneer work by Wigner has been refined as well as expanded by Dyson and Mehta [4, 5, 6, 7] and developed into a sound mathematical theory.

The principal idea is as simple as it is elegant. Physical problems are often solved by finding the eigenvalues of a matrix. For problems that are 'sufficiently' complex it is reasonable to assume that the matrix elements are random variables. Analysing

an ensemble of matrices leads then to a distribution of the respective eigenvalues and therefore to interesting physical properties. Wigner considered the choice of random variables as matrix elements as a way to represent our lack of knowledge about a highly complex system.

However, physical problems usually exhibit global symmetries, which have to be imposed on the matrix. These global symmetries restrict the configuration space of the eigenvalues leading to important changes in the distribution of the eigenvalues. To be more specific let us discuss a more explicit example from here on (another important example is discussed in the second chapter). The random matrix often represents the Hamiltonian H of a system, while its eigenvalues are the energy levels. As symmetry we can choose time reversal symmetry, which leads then, as well known from quantum mechanics, to a real, symmetric Hamiltonian. This reduces the actual number of independent variables and therefore independent matrix elements from N^2 to $N(N+1)/2$, whereas the configuration space of the eigenvalues is apparently only N -dimensional. The mapping from the space of independent matrix variables to the space of eigenvalues is therefore contractive and associated with a nontrivial Jacobian. In case that the matrix elements or random variables obey a standard gaussian distribution, one can calculate this Jacobian explicitly. This is instructive and will be outlined herein for the case of $N=2$. A general symmetric, real Hamiltonian takes then the form

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} \quad (1.1)$$

Diagonalized the matrix can be parametrized as

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \quad (1.2)$$

where E_1, E_2 represent the eigenvalues of the matrix H . Apparently a simple rotation can transform the Hamiltonian into its diagonal form $H = DED^T$, where D can be

parametrized by a single 'angle' Θ .

$$D = \begin{pmatrix} \cos(\Theta) & \sin(\Theta) \\ -\sin(\Theta) & \cos(\Theta) \end{pmatrix} \quad (1.3)$$

With these definitions one can easily evaluate the functional dependence of the matrix elements on the eigenvalues, which gives the Jacobian and consequently the invariant volume $dh_{11}dh_{12}dh_{22}$

$$dh_{11}dh_{12}dh_{22} = \det \begin{vmatrix} \frac{\partial h_{11}}{\partial E_1} & \frac{\partial h_{11}}{\partial E_2} & \frac{\partial h_{11}}{\partial \Theta} \\ \frac{\partial h_{12}}{\partial E_1} & \frac{\partial h_{12}}{\partial E_2} & \frac{\partial h_{12}}{\partial \Theta} \\ \frac{\partial h_{22}}{\partial E_1} & \frac{\partial h_{22}}{\partial E_2} & \frac{\partial h_{22}}{\partial \Theta} \end{vmatrix} dE_1 dE_2 d\Theta \quad (1.4)$$

resulting in

$$dh_{11}dh_{12}dh_{22} = |E_1 - E_2| dE_1 dE_2 d\Theta \quad (1.5)$$

This argument can be extended to N eigenvalues [4, 8] leading to $N(N-1)/2$ 'rotation angles' Θ_i .

$$dh_{11}..dh_{1N}dh_{21}..dh_{2N}..dh_{N1}..dh_{NN} = \quad (1.6)$$

$$F_1(\Theta_1, \dots, \Theta_{N(N+1)/2}) \prod_{i < j, i \neq j}^N |E_i - E_j| dE_1..dE_N d\Theta_1..d\Theta_{N(N+1)/2} \quad (1.7)$$

The function F_β (here $\beta = 1$) depends only on the 'rotation angles' Θ_i and has therefore no influence on the distribution of the eigenvalues E_i . However, the factor $\prod_{i < j, i \neq j} |E_i - E_j|$, which is often called the Jacobian factor, depends on the eigenvalues and will consequently affect the probability distribution of the eigenvalues. The most obvious feature of the Jacobian factor is that it vanishes at $E_i = E_j$ for $1 < i, j < N$. The probability for that to occur would therefore be of course zero, and the likelihood of two eigenvalues being very close to each other would obviously be quite small. This fundamental result is called level repulsion and has a profound impact on the behavior of the eigenvalues.

Note that the derivation has only assumed a matrix with random elements and a symmetry condition imposed on the matrix. The geometrical restriction from the space of independent parameters to the space of eigenvalues originates already from

the appearance of the Jacobian factor and therefore the level repulsion. This very general derivation shows the robustness and generality of the level repulsion.

The Jacobian factor, however, will be altered by the type of symmetry imposed on the matrix. In the example above a real symmetric matrix has been chosen, while in physics two more symmetries are relevant.

In general a matrix can also exhibit complex numbers as matrix elements with the hermiticity condition $H_{ij} = iH_{ji}$ being imposed on the matrix enforcing real eigenvalues. Then a unitary transformation U can be used to diagonalize the Hamiltonian. A hermitian matrix has $N + 2(\frac{1}{2})(N^2 - N) = N^2$ independent parameters, which are mapped onto the N -dimensional eigenvalue space. Consequently, the same rationale as above can be applied to obtain the invariant volume element

$$dh_{11}..dh_{1N}dh_{21}..dh_{2N}..dh_{N1}..dh_{NN} = \quad (1.8)$$

$$F_2(\Theta_1, \dots, \Theta_{N(N-1)}) \prod_{i < j, i \neq j}^N |E_i - E_j|^2 dE_1..dE_N d\Theta_1..d\Theta_{N(N-1)} \quad (1.9)$$

The Jacobian factor shows in case of unitary symmetry $\prod |E_i - E_j|^2$ with strong similarities to the case of orthogonormal matrices. Only the power of the difference coordinate is changed. It should not surprise us that the level repulsion is stronger for hermitian matrices because of the stronger restriction of the mapping from N^2 to N dimensions compared to $N(N+1)/2$ to N .

This result also appears in the last relevant symmetry, where the matrix cannot be represented by complex or real numbers but by quaternions. Physical examples will be outlined later in this chapter, however, it should be mentioned here that systems with spin $\frac{1}{2}$ and no rotational symmetry will be represented by this type of matrix. Even though the derivation for the invariant volume element is more involved, it is nevertheless straightforward and can be found in the literature [9, 4, 5] with the result

$$dh_{11}..dh_{1N}dh_{21}..dh_{2N}..dh_{N1}..dh_{NN} = \quad (1.10)$$

$$F_4(\Theta_1, \dots, \Theta_{N(2N-1)}) \prod_{i < j, i \neq j}^N |E_i - E_j|^4 dE_1..dE_N d\Theta_1..d\Theta_{N(2N-1)} \quad (1.11)$$

Again the Jacobian factor is only changed in the power of the difference variable which can be represented by the symmetry parameter β . By use of group theoretical arguments it can be shown that β can only take the values 1,2,4 depending on the symmetry case [4, 5]. Usually the three cases are called after the type of diagonalizing transformation necessary and the distribution of the random variables.

For real symmetric matrices the ensemble of matrices is therefore named Gaussian Orthogonal Ensemble (GOE), for hermitian matrices the ensemble is called Gaussian Unitary Ensemble (GUE) and for quaternion matrices the ensemble is known as Gaussian Symplectic Ensemble (GSE).

We would like to discuss the physical implications of these three basic ensembles. The one which is usually the easiest to treat is the gaussian unitary ensemble in which no symmetry except for the hermiticity of the Hamiltonian matrix is present. This situation is suitable for systems which are measured with a large applied magnetic field in order to break the time reversal symmetry. In case the time reversal symmetry is present in the system the analysis has to be further subdivided into the cases of integer and half-integer total spin. It is relatively easy to show that in the case of integer spin a basis can be chosen in which the matrix elements are real numbers [4]. The random matrix ensemble which is describing a real symmetric matrix is the GOE. This applies usually to systems which possess an integer spin and which are measured without a magnetic field present. The last case is intact time reversal symmetry and half-integer spin. The finding of an appropriate basis is slightly more involved and leads to 2x2 matrices as basic 'numbers', usually called quaternions. The resulting matrix is then a symplectic unitary matrix [4, 5], leading to the GSE.

To discuss the aspects of random matrix theory further we have to introduce a distribution function of the eigenvalues, which is laid out next.

1.1.1 The Joint Probability Distribution Function

The central quantity to completely describe a random matrix ensemble is the Joint Probability Distribution function (JPD). An elegant derivation of the JPD for random matrices can be realized by defining an Information Entropy analogously to the usual entropy in statistical mechanics. Maximization of the entropy with some given constraints results then in the JPD. This argumentation was first used by Balian [10] and Porter [9]. The information entropy is

$$I(P(H)) = \int dV P(H) \ln P(H) \quad (1.12)$$

Here $P(H)$ represents the probability distribution for the matrix H , while dV is the invariant volume element. The information entropy is also subject to the condition that the probability is normalized to one.

$$\int dV P(H) = 1 \quad (1.13)$$

One can easily see that this single constraint is not sufficient to define a suitable probability distribution. The level repulsion from the Jacobian would lead to a nonzero probability that eigenvalues are at infinity and, therefore, the distribution cannot be normalized. Hence another constraint is necessary to confine the eigenvalues around the origin. The usual choice of constraint is to postulate the following trace condition

$$\int dV \text{Tr}[HH^+] P(H) = \text{Const.} \quad (1.14)$$

This condition prevents very large eigenvalues in the random matrix restricted by the finite constant given. The choice of the trace of the matrix is a natural one, as the trace is invariant under all types of transformations in consideration, i.e. orthogonal, unitary and symplectic. It can be even shown that each invariant of a $N \times N$ matrix under a nonsingular similarity transformation can be expressed in terms of the first N powers of the trace [4, 11]. This assumption, however, is not the most general one in order to construct a random matrix ensemble. That point will be important in the

discussion of a generalized model in the second chapter.

The information entropy and the two constraints lead then to a maximization problem with two Lagrange multipliers λ, μ

$$I(P(H), \lambda, \mu) = \int dV \{P(H) \ln(P(H)) + \lambda P(H) + \mu \text{Tr}(HH^+)P(H)\} \quad (1.15)$$

The standard procedure is to set the first variation in $P(H)$ as a function of λ and μ equal to zero, resulting in

$$\delta I(P(H), \lambda, \mu) = \int dV \delta P(H) \{\ln(P(H)) + 1 + \lambda + \mu \text{Tr}(HH^+)\} = 0 \quad (1.16)$$

The choice of $\delta P(H)$ is arbitrary, and therefore the integrand has to be zero finally giving

$$P(H) = \exp(-(1 + \lambda + \mu \text{Tr}(HH^+))) \quad (1.17)$$

By expressing now the JPD in the diagonal representation with the Jacobian factor, noting that $\text{Tr}(HH^+) = \sum_i e_i^2$, and rescaling the energy variable $E_i = \sqrt{\mu} e_i$, we arrive at

$$P(E_i) = C_{N,\beta} \prod_{i < j}^N |E_i - E_j|^\beta \exp(-\sum_i E_i^2) \quad (1.18)$$

Here $C_{N,\beta}$ represents a normalization constant in order to satisfy Equation 1.13. Equation 1.18 is a fundamental result derived only with the assumptions of global matrix symmetry and statistical independence of linear independent components.

It is common to rewrite Equation 1.18 in the more suggestive form

$$P(E_i) = C_{N,\beta} \exp(-\beta \sum_{i < j} \ln(|E_i - E_j|) - \sum_i E_i^2) \quad (1.19)$$

The surprising fact is that the exponent exhibits the same behavior as the 2D-electron gas in a harmonic oscillator potential. This so-called Coulomb gas analogy is not only important for depicting a particular system, but also for applying known approximation techniques from the 2D-electron gas. Furthermore, it allows to speak of the eigenvalues as particles (electrons or charges) which is a useful and common analogy.

From equation 1.18 we can also introduce the terminology of the two body interaction or logarithmic repulsion represented by the first term in the exponent and the one body potential or confining potential which is the second term in the exponent. So far, the discussion has been limited to the distribution of eigenvalues, which is sufficient to find the probability distribution and its moments. However, the question can be extended to the eigenvectors of the system which contain information not available in the eigenvalue distribution alone. The eigenvector distribution is in particular important in cases where eigenlevel transition are triggered by external influence. An example is the description of nuclear reaction width. Also the description of localized or extended states in the theory of localization needs to know complete states. Porter [9] was the first to realize that it is possible to determine the distribution function for the n th component ($1 \ll n \ll N$), of an eigenvector. There result was

$$P(x) = \sqrt{\frac{1}{2\pi}} \frac{1}{\sqrt{x}} \exp(-x/2) dx \quad (1.20)$$

where $a_i^2 = Nx$ with a_i represents the n th component of the eigenvector. Naturally it is called Porter-Thomas distribution. One important quantity which uses the eigenvector distributions is the inverse participation ratio. It measures how extended the state is and presents a helpful tool in the theory of disordered conductors.

1.2 Correlation Functions and Spectral Statistics

1.2.1 Correlation Functions

The Joint Probability Distribution function contains all information about an ensemble. However, usually it is not necessary to know the entire distribution (which would be also quite cumbersome), since it is sufficient to concentrate on the lower moments of the distribution (or correlation functions), which contain the major physical aspects. A natural choice are the n -level correlation functions. These are defined as

$$R_N(x_1, \dots, x_N) = \frac{N!}{(N-n)!} \int \dots \int dx_{n+1} \dots dx_N P(x_1, \dots, x_N) \quad (1.21)$$

The function R_N describes the probability density that one eigenvalue is at x_1 , one at x_2 , up to x_N . The 1-level correlation function (also equivalent to the first moment) is the level density.

$$\sigma(x) = R_1(x) = N \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P(x, x_2, \dots, x_N) dx_2 \dots dx_N \quad (1.22)$$

Even though the n-level correlation functions have a direct probability meaning, for computational purposes it is useful to define separate correlation functions which are only nonzero when the n different levels are correlated, thereby eliminating level self-correlations. Such a set of functions is called Cluster functions and is usually defined as the logarithmic derivative of the n-level correlation function. We have first to define the generating function for the R_n

$$R_{\infty}(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} \left[\int \dots \int R_i(x_1, \dots, x_n) \prod_{j=1}^i y_j dx_j \right] \quad (1.23)$$

The generating function for the Cluster functions is similarly defined.

$$T_{\infty}(t) = \sum_{i=1}^{\infty} \frac{t^i}{i!} \left[\int \dots \int T_i(x_1, \dots, x_n) \prod_{j=1}^i y_j dx_j \right] \quad (1.24)$$

Now the Cluster functions are derived from the following relation

$$R_{\infty}(t) = \exp(T_{\infty}(t)) \quad (1.25)$$

resulting in

$$T_n(x_1, \dots, x_n) = \frac{1}{n!} \frac{\partial \ln R_{\infty}(1)}{\partial y_1 \dots \partial y_n} \Big|_{y_1=y_2=\dots=y_n=0} \quad (1.26)$$

The 1-level and 2-level correlation function in terms of Cluster functions are

$$R_1(x_1) = T_1(x_1) \quad (1.27)$$

$$R_2(x_1, x_2) = T_1(x_1)T_1(x_2) + T_2(x_1, x_2) \quad (1.28)$$

1.2.2 Statistical Analysis

The goal of statistical analysis is to provide a tool for comparing random matrix theories, analytical models, numerical calculations and experimental data on the statistical properties of a set of eigenvalues. Even though statistical analysis can usually not detect any specific physical aspects, it is quite useful to understand broad physical aspects of certain systems and to reveal universalities and qualitative features in complex systems.

Comparing the statistical properties of different systems requires to 'unfold' the spectrum. This procedure has to be followed by both the random matrix theory and the experimentally and numerically observed spectra to obtain comparable data. The goal is to fix the first moment properties of the spectrum, thereby attributing the second moment properties a universally comparable meaning. In the case of experimental or numerical data a level sequence $\{E_n\}_{1 \leq n \leq N}$ is given. Then one analyses the staircase function $N(E)$ of that set, where the staircase function is defined as the number of levels with an energy smaller than E . This function can be extended to a smooth approximating curve $\overline{N}(E)$ (for example by spline interpolation). Mapping the spectrum $\{E_n\}$ onto a new set of values $\{E'_n\}$ (the unfolded set of eigenvalues) by means of

$$E'_n = A[\overline{N}(E_n) - B] \quad (1.29)$$

(A, B are two arbitrary constants) results then in a constant level density, which becomes quite obvious from the straight line increase in $\overline{N}(E')$. The most common choice is to set the average spacing to 1. Now the fluctuations given by some second moment expressions provide the leading behavior.

The theoretical approach will usually also exhibit a nonuniform density. By means of the mapping

$$\sigma(E)dE = \sigma(E')dE' \equiv dE' \quad (1.30)$$

which requires the invariant volume to remain constant, one arrives by definition at $\sigma(E') \equiv 1$.

The nearest neighbour spacing distribution is defined by the probability to find the spacing ΔE between any two eigenvalues. As such it is a close range characteristic of the eigenvalue distribution. For a random matrix distribution the Wigner surmise for real symmetric matrices holds true.

$$P(x) = \frac{\pi x}{2D^2} \exp\left(-\frac{\pi x^2}{4D^2}\right) \quad (1.31)$$

D represents here the average spacing. Wigners argumentation [2] was based on the solvable 2×2 matrix system and then generalized for larger matrices. It should be emphasized that, despite the fact that 1.31 is not exact, it matches the exact function surprisingly well [4]. For all three ensemble types the spacing distribution reads

$$P_\beta(x) = A_\beta x^\beta \exp(-B_\beta x^2) \quad (1.32)$$

with A_β and B_β being constants. The slope at the zero argument is exact even for larger numbers of eigenvalues and, therefore, represents a possible way to distinguish the ensembles as well as distributions which differ from the three standard types, i.e. orthogonal, unitary and symplectic.

In considerable contrast to the Wigner surmise is the distribution of completely random levels, the Poisson statistics

$$P(x) = \frac{1}{D} \exp\left(-\frac{x}{D}\right) \quad (1.33)$$

The two distributions are plotted in figure 1.2.2 demonstrating the numerical ease to differentiate statistical signatures. This as well as the simple evaluation for numerical eigenvalues makes the nearest neighbour spacing distribution a popular tool for the statistical analysis.

The number variance is another important quantity for statistical analysis. It is defined as [4]

$$\langle n^2 \rangle = \int_{-L}^L \int_{-L}^L (R_1(x)\delta(x-x') + R_2(x, x')) dx dx' \quad (1.34)$$

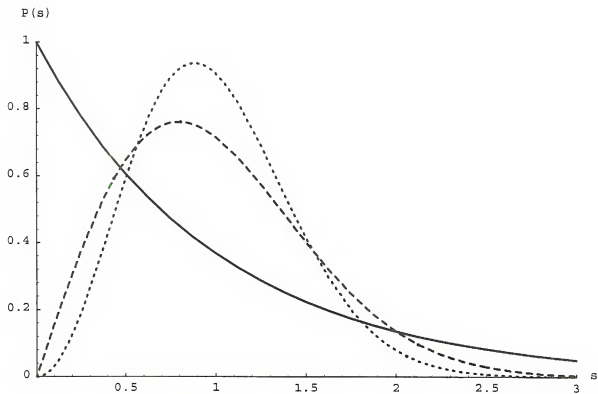


Figure 1.1: The nearest neighbour spacing distribution for Poisson and Wigner-Dyson ensembles in terms of the variable $s = x/D$, where x is the coordinate and D represents the mean level spacing. The dotted curve is $\beta = 2$ and the dashed one for $\beta = 1$.

where L is the range (support) on which eigenvalues are present. In terms of a fixed mean level spacing D , after the spectrum has been unfolded, the number variance reads

$$\langle n^2 \rangle - \langle n \rangle^2 = \int_{N/2}^{-N/2} \int_{N/2}^{-N/2} (\delta(x-x') - T_2(x, x')) dx dx' \quad (1.35)$$

It measures the statistical deviation of the eigenvalue system and involves as such only information of the first and second moment of the distribution. In contrast, the nearest neighbour spacing distribution contains information of all moments.

Another integral long-range measure of eigenvalue correlations is called Δ_3 -statistics. The Δ_3 -statistics was introduced by Dyson and Mehta [7] as their third Δ -statistics to measure the long-range statistical properties of eigenvalues for a finite set of eigenvalues. Only the Δ_3 -statistics proved to be a reasonable measure for eigenvalue rigidity. It measures the deviation from a totally rigid spectrum, that is, the deviation from an equally spaced eigenspectrum. More precisely, it measures the variations of the staircase function around a best fit line.

We will discuss in the following the derivation of a general expression for the Δ_3 -statistics, since this measure is not so widely known. Then we will calculate the Δ_3 -statistics for the limiting cases of Poisson and Wigner-Dyson statistics.

Assume we have a finite sequence of energy levels (they can come from experimental or numerical simulations). We can scale the eigenvalues onto an energy level sequence with constant density $\rho = \text{Const.}$ As discussed above with the appropriate rescaling, one can always achieve a unit density. The set of levels ranges without restriction from 0 to E_{tot} . The staircase function is defined as

$$N(E) = \int_0^E \rho(E') dE' \quad (1.36)$$

The variations around a straight line are given by

$$\Delta_3' = \min \overline{(N(E) - aE - b)^2} \quad (1.37)$$

$$= \min \frac{1}{n} \int_0^n dE' (N(E) - aE - b)^2 \quad (1.38)$$

The average denoted by the solid line takes a sufficient range of energy levels into account. The parameters a, b have to be chosen so that the extremal conditions are met. The conditions $\frac{d\Delta_3}{da} = 0 = \frac{d\Delta_3}{db}$ lead to $a = \frac{3}{4n^2} \overline{EN(E)}$ and $b = \overline{N(E)}$. Then

$$\Delta_3 = \overline{N^2} - \overline{N}^2 - \frac{3}{4n^2} \overline{EN(E)} \quad (1.39)$$

To obtain a statistically relevant measure we have to take a statistical average over an ensemble of eigenvalue configurations. Denoting $\langle \rangle$ as the ensemble average we have

$$\Delta_3 = \langle \Delta_3' \rangle \quad (1.40)$$

In terms of the density-density correlation function this can be expressed as

$$\Delta_3 = \frac{1}{n} \int_0^n d\tilde{E} \int_0^{\tilde{E}} dE \int_0^{\tilde{E}} dE' \langle \rho(E)\rho(E') \rangle \quad (1.41)$$

$$- \frac{1}{n^2} \int_0^n d\tilde{E} \int_0^n d\tilde{E} \int_0^{\tilde{E}} dE \int_0^{\tilde{E}} dE' \langle \rho(E)\rho(E') \rangle \quad (1.42)$$

$$- \frac{3}{4n^2} \int_0^n d\tilde{E} \int_0^n d\tilde{E} \int_0^{\tilde{E}} dE \int_0^{\tilde{E}} dE' \langle \rho(E)\rho(E') \rangle \quad (1.43)$$

The density-density correlation function can be calculated from the first two correlation functions. Using

$$\langle \rho(E)\rho(E') \rangle = \delta(E - E')\rho(E) + \rho(E)\rho(E') + Y_2(E, E') \quad (1.44)$$

and observing that $Y_2 = T_2/D^2$ one can analytically calculate the Δ_3 -statistics. For the Poisson case ($Y_2 \equiv 0$) we easily get

$$\Delta_{3, Poisson} = \frac{1}{15}n \quad (1.45)$$

For the GUE ($Y_2 = \frac{\sin^2(\pi x)}{(\pi x)^2}$) this can also be integrated in the large N limit (see [5])

$$\Delta_{3, Wigner-Dyson} = \frac{1}{2\pi^2} [\ln(2\pi n) + Const.] \quad (1.46)$$

We can already see the drastic difference between the linear increase in the Poisson case and the slow logarithmic rise for the Wigner-Dyson model, which so apparently

indicates the difference between a chaotic and a regular spectrum. The large N behavior of the Δ_3 -statistics is quite similar to the number variance, only at small N do these two statistics differ qualitatively.

1.3 Application to Physical Systems

The wide physical application of random matrix theory makes it almost impossible to provide a comprehensive coverage. Therefore, we will restrict ourselves to several important cases. More examples, pertaining to the transitional model at the core of this work, will be discussed later.

1.3.1 Nuclear Physics

As already mentioned, the first application of random matrices in physical systems was in nuclear physics. It was of particular importance to understand the excitation spectrum of nuclei, typically determined by slow neutron resonance measurements [12]. The lower lying energy levels are quite well understood by an independent particle model, where the nucleons move independently in an average potential [13] This approximation breaks down for higher lying excitations, and the subject becomes a full interacting N -body problem, where N is the sum of protons and neutrons. The high level of interaction makes the exact prediction of energy levels almost impossible. This was the reason for Wigner's proposal in 1955 [2] to accept the lack of information as an advantage and to assume the energy level distribution (at least locally) to be equal to the distribution of eigenvalues of a random matrix.

It is obviously quite difficult to prove the exactness of a random matrix approach, because specific models are usually neither solvable nor as general in nature as the random matrix approach. Therefore one has to resolve to numerical comparison and statistical analysis in order to show the accuracy of Wigner's assumptions. In the case of nucleonic behavior the two level properties (as for example number variance, Δ_3 -statistics) were in good agreement with the predictions from RMT (a good com-

prehensive reference is the book by Porter [9]). In contrast, the one level quantities like the constant level density showed less satisfactory agreement [5]. This represents one example where the statistical behavior demonstrates good agreement, whereas direct correlation functions still contain specific physical properties complicating a comparison with RMT.

1.3.2 Quantum Chaos

The first indication that random matrix theory is relevant for describing the spectral properties of chaotic systems resulted from studying the quantum stadium billiard [14]. A stadium billiard system is either integrable or chaotic. The authors showed that in the chaotic case (noncircular stadium) the level spacing was following a Wigner-Dyson distribution opposed to a Poisson distribution in the integrable case (circular stadium). Today we know that the dynamics of a classical system and the spectrum of its quantum analogue are tightly related. If the spectrum exhibits level repulsion, the classical system is chaotic. Due to this phenomenon random matrix theory plays an integral role in the understanding of chaos. The better understanding of the transition between these two limits has remained an outstanding topic. In classical systems constants of motion are destroyed by nonlinear resonances. The quantum counterpart is the destruction of good quantum numbers. As long as constants of motion (good quantum numbers) are present, degeneracies in the spectrum exist, which imply at least some Poissonian behavior. Classically chaotic systems can also exhibit many local features in their phase space (scars, KAM surfaces) which make the transition to a regular spectrum to some degree system dependent. Despite these possible objections, the search for a transitional theory represents an important topic in quantum chaos. An early phenomenological distribution by Brody [15] has often been used to fit numerical data of transitional systems. Another approach was based on the division of the phase space into chaotic and integrable regions [16], where

the distribution function was superposed from the (known) distributions in chaotic and integrable portions of the phase space. However, a large class of transitional systems seems to exist, which do not follow the characteristics of this transitional model. Further relevant and more recent models are described later in the discussion including the model presented herein.

1.3.3 Mesoscopic Physics

The rising interest in mesoscopic physics is another field, where random matrix theory can be applied to. A mesoscopic system is usually considered to range between macroscopic physics, where systems can be treated as self averaging, and the microscopic level, where fluctuations can be very large. In other words, the system is still large enough ($> 10^{-8}m$) to exhibit a bulk statistical behavior but small enough for fluctuations effects to occur ($< 10^{-5}m$). The usual quantity for defining a mesoscopic system size is the electron phase coherence length l_e .

One particularly interesting area in mesoscopic physics are disordered conductors, which exhibit a transition from metallic conductors to insulators depending on the disorder. The basic assumption here is that the quantum coherence length is larger than the mean free path of electrons, therefore the physical transport properties should be insensitive to microscopic details. The main quantity to be determined in the disordered conductor problem represents the conductance itself, which is defined by a whole conductance distribution $P(g)$. This can be seen from the fact that the conductance has to be a random variable, which fluctuates with each realization of the disorder potential. The usual transport theory is based on the Landauer formula, which describes the electrons to be in momentum channels, flowing from reservoir with chemical potential μ_1 to a reservoir with chemical potential μ_2 . The current depends then in the following way on the voltage

$$I = \left\{ \frac{e^2}{h} \sum_{i,j=1}^N T_{ij} \right\} V \quad (1.47)$$

where T represents the transmission matrix and $\frac{e^2}{h}$ the conductance quantum. Then we have for the conductance

$$g = \sum_{i,j=1}^{\infty} T_{ij} = TrTT^+ \quad (1.48)$$

This simplified approach was pioneered by Landauer [17, 18] and refined later in [19, 20]. The formula can be used to establish a scattering theory approach which relates ingoing flux into an interaction area to the outgoing flux. The scattering matrix is usually replaced by the transfer matrix M which relates fluxes on the right to the fluxes on the left.

$$M \begin{pmatrix} I \\ O \end{pmatrix} = \begin{pmatrix} I' \\ O' \end{pmatrix} \quad (1.49)$$

The current conservation law requires then the relation

$$|I|^2 - |O|^2 = |O'| - |I'|^2 \quad (1.50)$$

An appropriate choice of parametrization and the relation which connects the transfer and scattering matrix leads then by use of the Landauer formula to the conductance [21, 22, 23]

$$g = 2 \sum_{i=1}^N \frac{1}{1 + \lambda_i} \quad (1.51)$$

where the λ_i are eigenparameters of the parametrization chosen. This relation implies that the conductance is now given by an entire distribution, and mean conductance and conductance fluctuations involve now an average procedure over the distribution. The most striking feature in these systems is the appearance of universal conductance fluctuations ([24] and experimentally [25]), which results in the prediction of $\frac{2}{15}$ in terms of the dimensionless conductance. This quite fundamental phenomenon actually prompted the development of the theory mentioned above.

Quite active is also the area of ballistic quantum junctions. Advances in constructing microstructures make it possible to experimentally measure mesoscopic systems

in the ballistic regime. In these systems the scattering approach depicted before is usually applied in form of the scattering matrix. The junction is described by a Hamiltonian which is strongly coupled to the leads [26]. The appropriate ensemble for this configuration is Dyson's circular ensemble [27, 28]. It allows to calculate the variance of the conductance fluctuations (resulting in $\frac{1}{2\beta}$) as well as to predict the lineshape of the weak localization corrections [29]. It should be noted that the present random matrix theory can only account for chaotic junctions, whereas the nonchaotic junction in a random matrix formulation is still an open question. The difference in ensembles (from gaussian unitary/orthogonal to circular) originates from the fact that in disordered conductors the quasi one dimensionality is present in the multiplicative law 2.7, whereas ballistic micro junctions show no dimensional properties and are therefore sometimes regarded as 'zero-dimensional' systems.

This is quite similar to the ballistic quantum dot. Quantum dots separate the leads by tunnel junctions. The aspect of Coulomb charging effects is present here, which determines the spacing of the conductance resonances [30], but the resonance amplitude can still be calculated from random matrix theory [31].

Another interesting study area in mesoscopic physics are small metallic particles. These systems consist of an entire ensemble of approximately equal sized particles, usually mounted on a (insulating) matrix. This system can then be probed with external fields, and the response can be measured. The basic assumption here is that we practically have no knowledge about the particles, which makes the maximum randomness (and therefore random matrix theory) applicable. This field was pioneered by Gor'kov and Eliashberg [32] and has remained an active area (see [33]). Of interest here is that the experiments can also be repeated with more disordered metallic particles, which can also crossover to insulating behavior.

CHAPTER 2

THE TRANSITION MODEL

In the last chapter we discussed properties of standard random matrix theories, which are now an accepted method for determining features of chaotic systems and the metallic behavior of disordered conductors. This chapter introduces a generalized ensemble of random matrices, based on a maximum entropy ansatz, which is suitable for describing the transitional region of chaotic systems and disordered conductors.

2.1 Maximum Entropy Ansatz

We consider in the following a real symmetric random matrix H with matrix elements H_{ij} . In order to determine a probability density $P(H)$ over the space of matrices H , it is necessary to 'measure' the distance between matrices, that is, a metric has to be defined. The natural choice is

$$ds^2 \equiv Tr(dH)^2 = \sum_i (dH_{ii})^2 + 2 \sum_{i < j} (dH_{ij})^2 \quad (2.1)$$

The volume element is then given by

$$dH = \prod_i dH_{ii} \prod_{i < j} dH_{ij} \quad (2.2)$$

The metric and the volume element are invariant under real orthogonal transformations and the addition of a constant matrix.

The importance of these invariances can be exemplified by the similarity to the approach taken in differential geometry which, given a metric $g_{\nu\mu}$, defines a length element by

$$ds^2 = \sum_{\nu\mu} g_{\nu\mu}(x) dx_\nu dx_\mu \quad (2.3)$$

and volume element by

$$dV = |\det g(x)|^{1/2} \prod_{\nu} dx_{\nu} \quad (2.4)$$

Both measures are now invariant under the less abstract mappings of rotation and translation of the space $\{x_{\nu}\}$.

The following discussion is restricted to the orthogonal case ($\beta = 1$), but the generalization to the other cases (unitary and symplectic) are straightforward, as they only change the dimensional arguments. With the matrix measures we can write the differential probability as a product of the invariant measure and the probability density.

$$dP(\{H_{ij}\}) = p(\{H_{ij}\}) dH \quad (2.5)$$

The probability density expresses the probability that the set of matrix elements H_{ij} lies in the interval $[H_{ij}, H_{ij} + dH_{ij}]$. The parameter space is apparently $N(N-1)$ -dimensional (orthogonal example).

The probability density is now transformed to a N -dimensional parameter basis to span the space by independent components. As most obvious and usual choice of parameters serves the eigenvalue basis, while the radial eigenparameters of the transfer matrix approach to conductivity are another example. Lets call this N -dimensional parameter set $\{x_i\}$. The probability density can then be expressed as

$$dP(\{x_i\}) = J(\{x_i\}) w(x_i) \prod_i dx_i \quad (2.6)$$

The probability density $w(x_i) \prod_i dx_i$ gives the measure of the x_i in the N -dimensional parameter space, while the Jacobian $J(x_i)$ gives the measure of the x_i in the $N(N-1)$ -dimensional matrix space.

The non-compactness of the matrix space (which is simple to see by constructing a sequence of matrices which exhibits in the limit diverging matrix elements or eigenvalues) will make the integration over that space diverge.

This argument delineates why the weight function $w(x_i)$ has to decay for large x_i . Without it, the first principle $\int dP = 1$ cannot be obeyed. However, the weight factor $w(x_i)$ is not specified by the approach so far and has to be chosen appropriately. In contrast, given a certain symmetry of the matrix H and a unique set of parameters, the Jacobian factor $J(x_i)$ can uniquely be determined (except for a multiplicative constant). Therefore, the statistical properties have to be fixed by defining an additional condition on the probability density, which leads to the determination of $w(x_i)$.

In the derivation of the probability density in standard random matrix theory, the condition was to restrict the average trace of the matrix H . This definition is equivalent with the (rather natural) assumption that each linearly independent component is also statistically independent. The random matrix can then be imagined as containing matrix elements which obey a gaussian distribution and which lack any correlation between each other (except, of course, the symmetry requirement).

Furthermore, the trace depends only on the N -eigenparameters. This property ensures the zero parameter dependence of standard random matrix theory (after the average eigenparameter separation has been scaled out). Certainly, this assumption has already physical relevance, but it is restrictive for other physically motivated matrix element correlations [34, 35].

Therefore, the goal is to find a more general way of fixing the statistical properties of the random matrix approach. The Jacobian or interaction term $J(x_i)$ provides already the level repulsion, but leaves the eigenparameter density $\sigma(x)$ unspecified. This leads to the approach to fix the weight factor $w(x_i)$ by defining the density $\sigma(x)$.

This idea is supported by the results from experimental and numerical studies. Often the density (as a first moment of the distribution P) shows system dependency [36, 86, 37], whereas the variance (or second moment of P) displays universal behavior.

Since a general theory like the random matrix approach is most suited for describing universal behavior, it is plausible to take the system-dependent density as

an input parameter to determine the weight factor $w(x_i)$ (and therefore the confining potential). Due to the global character of the density this ansatz is often called global maximum entropy approach (see [21, 22])

For transfer matrices M a different kind of approach is possible. It originates from the multiplicity law transfer matrices obey. Consider a system (a mesoscopic wire for example) of length L with the goal to add a small piece δL , which is still of mesoscopic size. The associated transfer matrices are called M_L and $M_{\delta L}$. Then the resulting transfer matrix is

$$M \equiv M_{L+\delta L} = M_{\delta L} M_L \quad (2.7)$$

If the two pieces are statistically independent, the probability density for the combined block can be written

$$p_{L+\delta L}(M) = \int p_L(M_{\delta L}^{-1} M) p_{\delta L}(M_{\delta L}) dM_{\delta L} \quad (2.8)$$

The measure dM is the invariant measure associated with the transfer matrix ensemble. A similar integral equation appears also in Markovian processes and allows the conversion into a Fokker-Planck type differential equation. The physical restriction necessary to fix the statistical properties is usually chosen to assign a mean free path value 1 [39, 40] This restriction leads to an isotropic probability density, which is also preserved in equation 2.8. Then one obtains for $\delta L \rightarrow 0$ []

$$\frac{\partial w_s^{(\beta)}(\lambda)}{\partial s} = \frac{2}{\beta N + 2 - \beta} \sum_{a=1}^N \frac{\partial}{\partial \lambda_a} [\lambda_a (1 + \lambda_a) J_\beta(\lambda)] \frac{\partial}{\partial \lambda_a} \frac{w_s^{(\beta)}(\lambda)}{J_\beta(\lambda)} \quad (2.9)$$

where $s = \frac{L}{l}$ and the initial condition is given by

$$w_0^{(\beta)} = \delta(\lambda) \quad (2.10)$$

This equation has recently been exactly solved [41]. The solution shows a small contribution from a second two body interaction term, which makes the global approach for transfer matrices just an approximation to the problem, albeit an excellent one,

as a second two body interaction term, is not present in the global formulation. The same problem does not arise for the eigenvalues of a Hamiltonian matrix, since the energy levels are directly the eigenvalues of the Hamiltonian.

However, it is important to mention that the solution of 2.9 has a potential which displays a lognormal dependence for large arguments. This is an important indication on how a model in global maximum entropy framework should be constructed.

2.2 Formulation

2.2.1 Generalized One Body Potential

The maximum entropy ansatz determines the weight factor from the density. Furthermore, the weight factor can be determined by the Lagrange multiplier method already used in 1.15. In general one has the functional integral

$$S[w(\{x_i\})] = - \int w(\{x_i\}) \ln(w(\{x_i\})) d\{x_i\} \quad (2.11)$$

subject to the constraining conditions

$$\langle f_n(\{x_i\}) \rangle = a_n \quad (2.12)$$

Here i runs from zero to the number of constraints and $\langle \rangle$ represents the expectation value

$$\langle f(\{x_i\}) \rangle = \frac{\int \prod_i^N dx_i f(\{x_i\}) P_\beta(\{x_i\})}{\int \prod_i^N dx_i P_\beta(\{x_i\})} \quad (2.13)$$

The comparison with the given density

$$\sigma(x) = \langle \sum_i^N \delta(x_i - x) \rangle \quad (2.14)$$

gives immediately an infinite number of constraints defined by a function $n \rightarrow x, a_n \rightarrow V(x)$. Lagrange's procedure results then in the variation

$$\delta S = - \int dx \delta p(x) \ln[p(x)] + 1 - \int V(x) \sum_i \delta(x - x_i) dx = 0 \quad (2.15)$$

obtaining

$$w(x_i) = \exp\left(-\sum_{i=1}^N V(x_i)\right) \quad (2.16)$$

Lets now concentrate on the generic form of the Jacobian in equations 1.6,1.8,1.10, where we can write (again including the symmetry factor β)

$$P(x_i) = C_{\beta,N} \exp(-\beta H) \\ C_{\beta,N} \exp\left(-\beta \sum_{i < j}^N \ln(|x_i - x_j|) + \sum_i^N V(x_i)\right) \quad (2.17)$$

This is the fundamental result for the JPD from the maximum entropy ansatz. The power law dependence of the Wigner-Dyson ensemble is replaced by an arbitrary function V , which is determined by the density. The generalization of Wigner's result 1.18 is obvious. In addition, the potential V allows for parameter dependence, a possibility which is not present in the original standard model.

The potential can be approximated in the continuous, large N limit by a mean field argument, which was originally used by Wigner to calculate the density of the standard random matrix model (the famous semi-circle law).

The most probable configuration of the eigenvalues $\{x_i\}$ can be found by calculating $\frac{\partial H}{\partial x_i}$ which results in

$$-\frac{\partial V}{\partial x_i} + \sum_{i \neq j}^N \frac{1}{|x_i - x_j|} = 0 \quad (2.18)$$

By ignoring the 'self-energy' of each eigenvalue (the eigenvalue self-correlation), and assuming the average and most probable configuration to be sufficiently close, the potential can be estimated in the mean field sense as:

$$V(x_i) = \int \sigma(x') \ln(|x - x'|) dx' \quad (2.19)$$

By using the functional derivative

$$\langle \overline{\sigma}(x) \overline{\sigma}(x') \rangle - \langle \overline{\sigma}(x) \rangle \langle \overline{\sigma}(x') \rangle = -\frac{1}{\beta} \frac{\delta \sigma(E)}{\delta V(E')} \quad (2.20)$$

it can be elegantly demonstrated that the variance of any linear statistic

$$F = \int f(x) \bar{\sigma}(x) dx \quad (2.21)$$

($\bar{\sigma}(x) = \langle \sum_i \delta(x_i - x) \rangle$ is the microscopic density) is independent of the potential V and therefore universal [42]. This underlines the success of the random matrix theory constructed by means of the global maximum entropy approach.

It is important to recognize the limitations of the result above. A necessary condition has been the ability to go to the continuous limit in the case $N \rightarrow \infty$. This, however, depends on the strength of the confining potential. A 'strongly' confining potential will limit the eigenvalues close to the origin, and therefore in the $N \rightarrow \infty$ limit the eigenvalues will move very close together allowing a continuum approximation. In case of a 'weak' potential, where the eigenvalues move towards infinity, the continuum reasoning does not apply. The role of weak potentials represents a central topic in this investigation and will be pursued below.

2.2.2 The Method of Orthogonal Polynomials

The equilibrium probability density function 2.17 requires apparently a more general formalism than the continuum limit approximation, which allows to cover arbitrary potentials and a finite number of eigenvalues. This formalism is the method of orthogonal polynomials and was first used by Mehta [4]. The key observation consists of the fact that the Jacobian factor $\prod_{i,j}^N |x_i - x_j|^2$ ($\beta = 2$ for simplicity) can be written in terms of a Vandermonde determinant

$$\prod_{i,j}^N |x_i - x_j|^2 = \left| \begin{array}{cccc} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \dots & \dots & \dots & \dots \\ x_1^N & x_2^N & \dots & x_N^N \end{array} \right|^2 \quad (2.22)$$

By using properties of determinants one can multiply the $(j-1)$ th row with an arbitrary factor and add it to the j th row. Continuing this procedure to the N th row leads to a polynomial of the j th degree in the j th row. Multiplying each row with the potential

term results in

$$\begin{aligned} P(\{x_i\}) &= \prod_{i < j}^N |x_i - x_j|^2 e^{-\sum_i^N V(x_i)} \\ &= \text{Adet}(k_j(x_i) \exp(-\frac{1}{2}V(x_i))) \end{aligned} \quad (2.23)$$

where $k_j(x_i)$ represents a polynomial of degree j . The advantage of choosing the orthogonal polynomial set $r_j(x_i)$ which corresponds to the integration measure $\exp(-\frac{1}{2}V(x))dx$ for the polynomial $k_j(x_i)$ is obvious, as it allows to evaluate n -point correlation functions by using the orthogonality of the polynomials

$$\int_{-\infty}^{\infty} r_i(x) r_j(x) \exp(-\frac{1}{2}V(x)) dx = s_n \delta_{ij} \quad (2.24)$$

(s_n is the normalization of the polynomials r_i). The density (one point correlation function) for example is given by

$$\begin{aligned} \sigma(E) &= N \int_{-\infty}^{\infty} dx_2 \dots dx_N P(x_i) \\ &= \sum_{n=0}^{N-1} r_n^2(x) \exp(-V(x)) \end{aligned} \quad (2.25)$$

and the two point correlation function is

$$\begin{aligned} R_2(x, x') &= N(N-1) \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} dx_3 \dots dx_N P(x_1, x_2, \dots, x_N) \\ &= \sum_{i=0}^N r_i^2(x) \exp(-\frac{1}{2}V(x)) \sum_{j=0}^N r_j^2(x') \exp(-\frac{1}{2}V(x')) - \\ &\quad (\sum_{i=0}^N r_i(x) r_j(x') \exp(-\frac{1}{2}V(x) - \frac{1}{2}V(x')))^2 \end{aligned} \quad (2.26)$$

As a simple example, the standard random matrix model has the confining potential $V(x) = x^2$, which leads naturally to the Hermite polynomials $r_i(x) = H_i(x)$ as orthogonal polynomial set.

The importance of this method lies in the generality in which a potential $V(x)$ can be treated. For each integration measure $w(x)dx$ an orthogonal polynomial set exists, making the solution of the joint probability density 2.17 for each $V(x)$ in principle

accessible. However, for practical purposes it is better to use a potential which has a well-known and recorded associated polynomial set.

2.3 Formulation of a Spectral Transition Model

2.3.1 Strongly and Weakly Confining Potentials

The maximum entropy ansatz leads to the relatively independent choice of the confining potential, which is only fixed by the density configuration. Remember that this additional freedom was not present in standard random matrix theory. The limitation of the trace of a random matrix is straightforward leading to a harmonic oscillator confining potential.

However, the trace restriction is implied by statistical independence of the matrix elements. They obey the symmetry condition, but otherwise they are chosen as randomly as possible. This 'maximum randomness' approach is quite useful for describing physical systems from which we have minimal information such as completely chaotic systems.

However, there are many systems which do not belong to this class. The matrix elements usually contain further correlations, rendering their evaluation more difficult. The global maximum entropy ansatz allows to use a measurable quantity like the density and to construct then a reasonable random matrix theory.

As already mentioned before, the confining potential can fall into two classes: weakly and strongly confining. By imagining the eigenvalues of the JPD in 2.17 as two dimensional electrons or charges which repel each other with the logarithmic two body potential, we can define the difference between these two confining potential types. In the strongly confined case the potential is approximately given by $V(x) = \int \sigma(x') \ln(|x - x'|) dx'$. The electron density σ has to be neutralized by the potential V , otherwise the electrons would move. Therefore the interpretation of the potential as to arise from a neutralizing background grid of positive charges is feasible. This grid also allows the electron to define a continuous density, even though

the electrons repel each other. In the case that an additional electron is added to a system of N electrons, an additional positive charge has to be added, too. This charge permits the electron system to be compact and not to expand the energy coordinates to infinity.

The situation is different in the case of a weakly confining potential. It cannot be interpreted as a neutralizing background charge, because the electron system will become larger in the sense that the difference between the electrons with the smallest and the largest coordinate grows by a fixed (N -independent) amount. For infinitely many electrons the position coordinate will also go towards infinity. This rationale allows to define weak and strong potentials, depending on whether

$$\lim_{N \rightarrow \infty} \langle \text{Max}(\{x_i\}) \rangle > \quad (2.27)$$

is a finite limit or not. A local version of this condition is

$$f(x_i) = \lim_{N \rightarrow \infty} \langle x_i - x_{i+1} \rangle \quad (2.28)$$

going to zero or not. The finite distance between charges makes a continuous description, as in 2.20, not exact for $N \rightarrow \infty$. This is why the orthogonal polynomial formalism, which is explicitly N -dependent, is more appropriate for the desired transition model.

It is also this property which will restrict the validity of the universality of any linear statistics in equation 2.21 to cases where a continuous limit can be taken in an appropriate manner. The functional derivative reasoning will only apply for a continuous spectrum.

There have been already investigations concerning the question how strong the long distance decay behavior of the potential has to be in order to exhibit transitional properties. They indicate that all potentials decaying slower than $V(x) \propto x^{1/2}$ have a different qualitative behavior.

2.3.2 Requirements

The method of orthogonal polynomials enables to solve for the solution of equation 2.17 for all potentials $V(x)$, which does not violate the normalization requirement $\int P(\{x_i\})d\{x_i\} = \text{finite}$. To construct a model which displays transitional properties in its spectral statistics therefore means to choose the potential V appropriately.

For simplicity, the model should depend solely on one parameter. This is an obvious choice, since the main objective of the discussed model is to capture the qualitative features of a transition model. For physical purposes this parameter can be regarded as a measure of disorder or the transition parameter for chaotic systems. However, one has to realize that an exact identification of this model parameter with specific theoretical models is difficult due to the semiphenomenological nature of this model. Even more important, the desired model should pass over to the two known limits, the Wigner-Dyson random matrix model and the Poisson statistical model. This requires the model to have a weakly confining potential at large arguments. A particularly good choice seems to be the lognormal potential $V \propto \ln^2(x)$. This potential in the disordered problem occurred first in the one dimensional transfer matrix approach by Melnikov [43, 44]. Even though this result is not directly comparable with the description of disorder phenomena via energy level correlations, it still represents a cause to investigate the corresponding transfer matrix related q-Laguerre model [45, 46]. This model, which apparently belongs to a new universality class, confirms the unique nature of the weakly confining potential including translational symmetry breaking.

As this might already be considered sufficient motivation to investigate an energy level model containing a weakly confining potential, even stronger evidence of its relevance comes from several numerical studies [22, 47, 48, 49], which compare a similar model with specific disordered or chaotic systems. Here the 3-dimensional tight binding Anderson model should in particular be mentioned. Canali [47] shows quantitative

agreement between a model with weakly confining potential and the 3D Anderson model at the critical point. For example the linear part of the number variance and the nearest neighbour spacing distribution show good agreement.

The standard random matrix models incorporate a strongly confining potential. The transition potential should have a similar behavior for a certain value of the parameter. While the weakly confining part has to occur at large arguments x , the Wigner-Dyson-like strongly confining part of the potential arises at small arguments x . Appropriate for describing energy levels, the limiting potential has a square dependence. This type of model can be formulated in terms of orthogonal polynomials with the Hermite ensemble. Therefore the other limit of the desired potential is

$$\lim V(x \rightarrow 0) \rightarrow x^2 \quad (2.29)$$

It is obvious that this choice has to depend on a parameter in order to display a change in statistical behavior. The potential furthermore requires the recovery of the Wigner-Dyson limit, for example a square dependence for the Hamiltonian model.

2.4 Transition Ensembles

After having formulated the conditions on a transition model, an appropriate type of potential has to be chosen. The problem consists in finding a potential which corresponds to a well studied polynomial set allowing the use of the method of orthogonal polynomials. Furthermore, this potential should depend on one parameter. The first approach in this respect was proposed in [50] with the potential being

$$V(x) = -a \ln^2(1 + bx) \quad (2.30)$$

It should be noted that this ansatz has two parameters, whereas the other two conditions are certainly satisfied. This ansatz was justified by conducting a numerical analysis to the tight binding Anderson model. The result was a quite reasonable agreement in the correlation functions [50]. From this perspective it was natural to

investigate the q -Laguerre ensemble [45, 46] as an attempt to model the transitional properties of transfer matrices, which constitutes a direct model for the transport properties of disordered conductors. As mentioned before, this is an exact solution which allows precise predictions from the formulation. The one body potential is linear in x for small x and shows a lognormal dependence for large x .

The properties of this model exhibited a decay behavior in the density for a larger disorder parameter and a rapidly changing density close to the origin. In terms of the conductance, the model was found to be in qualitative agreement with the disordered conductor physics, i.e. the dependence on the disorder parameter exhibited both metallic and insulating behavior.

However, the small discrepancy between this model formulated in the global entropy assumption and the results from the local entropy ansatz (this becomes most clear from the occurrence of an additional, albeit small, two body interaction term) makes a further quantitative agreement disputable.

Another aspect of transitional models is the breaking of the invariance under unitary transformations or $U(N)$ -symmetry. This symmetry is inherent to standard random matrices, but any transitional statistics will have to break the unitarity. This is immediately obvious from the fact that the standard random matrices have no inherent scale other than the number of eigenvalues. In disordered conductors, for example, the standard random matrix theory describes well the eigenstates of the metallic phase, which are all extended. In this phase one set of eigenstates describes the whole system completely, i.e. every other eigenstate set can be found by linear combination. In the insulating phase the eigenstates are localized and therefore of completely different nature. Apparently, the 'featureless' standard random matrix theory cannot describe this fundamental transition.

This aspect was pointed out in [53, 54] along with the argument that this symmetry breaking is spontaneous in weakly confining potentials. Other approaches recognize

the symmetry breaking directly, by defining a random matrix theory, whose $U(N)$ -symmetry is broken by construction. This approach was favored in reference [55], where JPD had the following form:

$$P(H) = C_n \exp(-\text{Tr} H^2 - h^2 N^2 \text{Tr}([\Lambda, H][\Lambda, H]^\dagger)) \quad (2.31)$$

The model introduces explicitly an unitarity breaking term, which depends on one parameter h , measuring how strongly the symmetry is broken. The matrix Λ , which itself is unitary, indicates a preferred direction in matrix space for the Hamiltonian. The idea behind this random matrix model with preferential basis is to average over the matrix Λ . The model has the corresponding Wigner-Dyson limit for $h = 0$. The interesting result from this theory is that many features of the preferential basis model are very similar to the weak confinement model treated in here. In particular, the two-level correlation function (for $hN^2 = \text{const.}$) in the critical regime is identical to the one in the transitional model treated in here and derived in the next chapter. This relation will be evaluated in more detail in the discussion.

Another way of breaking the symmetry is to introduce banded random matrices. The physical intuition behind this model is the notion that the further the matrix element is away from the main diagonal, the smaller it becomes. A matrix element on the main diagonal can be interpreted in terms of the degree of the overlap between the state, in this case, with itself. These models have also one parameter, commonly called matrix bandwidth b , which measures how fast the matrix elements decay with increasing distance from the main diagonal. The transitional parameter is usually cited as b/N , with N being the number of eigenvalues [35, 56, 58, 61]. So far, most investigations point into the direction of this model being most feasible in quasi-1D [59, 60].

2.5 Applicability to Physical Systems

One possible application of a transition random matrix theory is to understand the energy level behavior in disordered systems. An example system are small metallic mesoscopic particles.

It would represent a particularly important step, if the metal-insulator transition was better understood in terms of a random matrix formulation. In the metallic regime the Wigner-Dyson random matrix theory applies. The system is in its diffusive state, where electrons lose their memory on the scale of the mean free path. As long as the mean free path is smaller than the sample size L , the electrons are diffusing over time through the entire sample. This 'ergodicity' defines a sample characteristic time and energy scale, t_c and E_c , where the energy scale E_c is usually called Thouless energy. The Thouless picture connects the static energy spectrum of the object with a transport property. However, this picture breaks down, when the second energy scale, the meanlevel spacing, becomes comparable to the Thouless energy. When the diffusive regime is crossing over to the ballistic regime, the electrons will also experience the sample edges (boundary) and therefore will know about the dimensionality. The usual standard random matrix ensembles do not provide any information about the dimension. This makes them only suitable for describing the ergodic metallic regime [57]. From this discussion it becomes apparent that a new type of random matrix theory is required for describing the statistical properties of disordered systems. The interest has increased recently, as it was suspected that at the critical point in 3D the statistical fluctuations are scale invariant. This led to speculations that the behavior at the critical point is governed by a unique statistics, i.e. neither Wigner-Dyson nor Poisson [63, 64]. This renders the study of the transition model very interesting.

Chaotic systems represent another area of physical interest. The term chaotic systems should be defined more precisely. We are usually interested in quantum systems whose classical counterpart is chaotic. This correspondence is necessary,

because the definition of chaos in quantum systems is not straightforward, whereas the Ljapunov exponents give a precise meaning to classical chaos. A regular classical system describes the particles as moving in phase space on invariant tori [65] in closed orbits. The transition to chaos is then seen by the breakup of the invariant tori by nonlinear resonances [8]. This transition as a function of a parameter will initially start only in parts of the phase space and will gradually proliferate over the entire phase space to complete chaos. On this basis we can understand the terms soft and hard (complete) chaos.

The quantum picture is more complex, as particles can tunnel from their present state into another state rendering the picture above inaccurate. A pragmatic, albeit somehow nonquantitative, approach to quantum chaos uses the statistical properties as a measure of how chaotic the system is. As already mentioned, the statistical properties of regular and chaotic systems show drastically different behavior. The statistical properties are also easily available for both numerical and experimental sources.

Standard random matrix theory has been very successful in explaining the statistical properties of chaotic systems [66, 8], while directly measurable quantities have been more difficult to bring into accordance with random matrix theory. An example is the energy level density in heavy nuclei, which is not as flat as the gaussian orthogonal predicts [9].

The large number of transitional systems makes a theory, which is capable of predicting the transition from regular to chaotic behavior, desirable. Some systems, like billiards for example, are either integrable (regular) or nonintegrable (chaotic). The transition between these two limits is usually abrupt, and a parameter which determines this transition is not always obvious. Furthermore, billiards usually exhibit one or more spatial characteristic length scales, the shortest periodic orbit being most important.

Such peculiarities in a system are usually difficult to explain by random matrix theories. But others, for example the hydrogen atom in a strong magnetic field, show transitional behavior as function of a parameter (the magnetic field). It can experimentally be generated with relative ease, and the Hamiltonian equations can be analyzed by numerical means. Its statistical behavior is very similar to the weakly confining potential theory [67]. For this class of systems it is quite important to develop a theory which can characterize the transitional features of chaotic systems. The complexity of analytic results of chaotic systems makes the comparison of the statistical behavior an essential part.

Another class of systems, which could be better understood by looking at the transitional model, are quantum systems depending on external parameters [68, 69, 70, 71, 72, 73]. The external parameter is physically quite different from the transitional parameter discussed before. The transitional parameter is connected to the disorder in the system and represents an internal physical quantity, which is fixed for a given system. One needs an entire ensemble of systems in order to get meaningful (ensemble averaged or disordered averaged) results. In contrast, the external parameter can be varied from the outside and for each given system independently. There are quite many systems, which fall into this category. Just a few examples are mesoscopic rings with internal magnetic flux, mesoscopic systems with external electric or magnetic field and also quantum dots not in complete chaos with the presence of an external field.

CHAPTER 3

THE DEFORMED GAUSSIAN ENSEMBLE

As outlined before, this chapter investigates the model based on the ensemble of q -deformed Hermite polynomials or, in short, q -Hermite polynomials. This model is particularly interesting, since it provides additional insights for the behavior of energy levels in disordered and chaotic systems.

3.1 Formulation

The general theory, described in the second chapter, allows to define a unique random matrix model by specifying an orthogonal polynomial system. Given a complete polynomial system, a weight function (or invariant measure) can be found which corresponds to the one body, confining potential. In case of the q -Hermite ensemble the potential is given by

$$w(x) = \frac{1}{(-q \exp(-2x); q)_{\infty}} \frac{1}{-q \exp(2x); q)_{\infty}} \quad (3.1)$$

$$= \frac{1}{(-q \exp(2x), -q \exp(-2x); q)_{\infty}} \quad (3.2)$$

where

$$(a; q)_{\infty} = \prod_{i=1}^{\infty} (1 - aq^i) \quad (3.3)$$

or, in general,

$$(a_1, a_2, \dots, a_n; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_n; q)_{\infty} \quad (3.4)$$

The formulation in this chapter is expressed in terms of the general variable x . For physical interpretation the variable x can be replaced by the energy coordinate E .

In order to ensure the suitability of this potential for the description of transition

energy level statistics, three conditions have to be met. The first one, the dependence on one parameter, is immediately obvious.

However, the other conditions have to be checked explicitly, with regard to a standard square dependence of the potential at small coordinate x and a lognormal dependence of the potential at large coordinate. Jacobi's triple product formula provides one way to verify this.

$$(q^2; q^2)_\infty (-zq; q^2)_\infty \left(-\frac{q}{z}; q^2\right)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \quad (3.5)$$

Choosing $z = e^{2x}$ we arrive at

$$\frac{1}{w(x)} \approx \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2} e^{2xn} \quad (3.6)$$

The sum can be approximated by an integral. Completing the square in the exponent results in

$$\frac{1}{w(x)} = \int_{-\infty}^{\infty} e^{-\gamma n^2 + 2xn} \quad (3.7)$$

$$= \frac{1}{\sqrt{2\pi\gamma}} e^{x^2/\gamma} \quad (3.8)$$

where γ is related to q by $q = e^{-\gamma}$. Scaling the coordinate ($x \rightarrow \sqrt{\gamma}x$) is eliminating the problematic $\gamma \rightarrow 0$ limit. The joint probability distribution function is then given by

$$P(\{x_n\}) = |x_i - x_j|^2 w(x) \quad (3.9)$$

A transformation to variable $\sinh x = \chi$ gives

$$w(\chi) = J(x, \chi) \sqrt{2\pi\gamma} e^{-\text{arcsinh}^2(x)} \quad (3.10)$$

where the Jacobian $J(x, \chi)$ has to be independent of the potential. Now the expansion of the arcsinh function for small and large argument χ shows the desired requirements. Therefore, the q -Hermite ensemble is a suitable choice for a transition statistics model. The general theory allows to write the JPD as

$$P(x_i) = \det[\phi_i(x_k)]^T [\phi_k(x_j)] \quad (3.11)$$

with the $\phi_i(x_j)$ representing the eigenfunctions ('single level wave functions') of the model given by

$$\phi_n(x_m) = \frac{1}{k_n} h_n(x_m; q) \sqrt{w(x)} \quad (3.12)$$

where the polynomials $h_n(x_m)$ are defined by the recursion relation

$$h_{n+1}(x; q) = 2x h_n(x; q) - \frac{1 - q^n}{q^n} h_{n-1}(x; q) \quad (3.13)$$

and the relations

$$h_0(x; q) = 1 \quad (3.14)$$

$$h_1(x; q) = 2x \quad (3.15)$$

The k_n are the normalization factors given by the orthogonality relation [76]

$$\int_{-\infty}^{\infty} h_n(x; q) h_m(x; q) w(x) dx = \gamma(q; q)_{\infty} (q; q)_n q^{-n(n-1)/2} \delta_{nm} \quad (3.16)$$

$$\equiv k_n \delta_{nm} \quad (3.17)$$

These equations define the q -Hermite random matrix ensemble.

In the mathematical literature these polynomials are called the q^{-1} -Hermite polynomials which were widely investigated [74, 75]. There are actually infinite possibilities to generalize the q -Hermite polynomials. However, it is important to note that even though the generalization is not unique, the displayed qualitative behaviour of each ensemble after rescaling is the same. This rescaling with the density is necessary, as can be seen from the the difference in the level density between q -Hermite and scaled q -Hermite polynomials, which we will discuss later.

3.2 Correlation Functions

3.2.1 Level Density

The eigenvalue level density is then easily expressed in the form

$$\sigma_N(x; q) = \sum_{n=0}^{N-1} \phi_n^2(x) \quad (3.18)$$

$$= \frac{1}{k_n} \sum_{n=0}^{N-1} h_n^2(x) w(x) \quad (3.19)$$

In the case $N \rightarrow \infty$ the density can be calculated in closed form (compare the calculation of the kernel) resulting in

$$\sigma_{\infty}(x; q) = \frac{1}{2\gamma} \quad (3.20)$$

Here

$$\gamma = \ln(1/q) \quad (3.21)$$

is a convenient redefinition of the deformation value q . For finite N the density has to obey the sum rule $\int \sigma_N dx = N$, and therefore the density has to vanish for large x . This is confirmed by numerical evaluation shown in figure 3.1. Visible is the typical behaviour of a relatively constant density close to the origin. When the coordinate x approaches N , the integral displays a relatively sharp drop to zero. The finite case is discussed in more detail later in this chapter. However, the value of the density at the origin varies as a function of q . Apparently the limit $q \rightarrow 1$ cannot be obtained with these polynomials (as the density diverges), but rescaling of the polynomials h_n (and consequently the potential w) into the $\bar{h}_n$ provides a well defined density limit.

$$\gamma^{n/2} \bar{h}_n(x; q) = h_n(\sqrt{\gamma}x; q) \quad (3.22)$$

$$\overline{w(x)} = w(\sqrt{\gamma}x) \quad (3.23)$$

Alternatively one can use the modified recursion relation

$$\bar{h}_{n+1}(x; q) = 2x\bar{h}_n(x; q) - \frac{1 - q^n}{\gamma q^n} \bar{h}_{n-1}(x; q) \quad (3.24)$$

The results for the density of the scaled polynomials $w(\sqrt{\gamma}x) \sum_n^N \bar{h}_n^2(x; q)$ are displayed for finite N in figure 3.2. Here the smooth crossover from the q -Hermite ensemble ($q < 1$) to the Hermite ensemble is obvious, showing Wigners semi-circle law for finite N .

The slower decay of the density can be attributed to the weaker potential. A strongly confining potential (for example the usual square potential) keeps the eigenvalues close to the origin. However, in a weakly confining potential 3.7 the eigenvalue

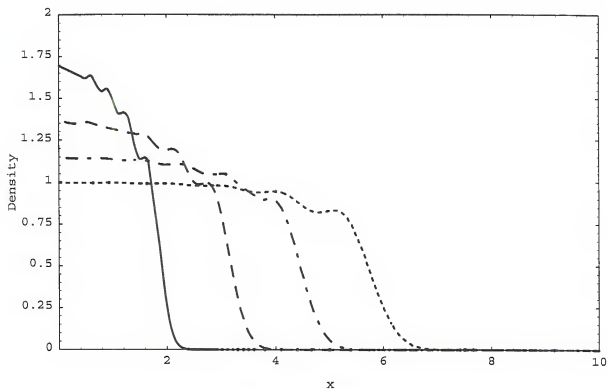


Figure 3.1: Eigenvalue density for the unscaled q -Hermite ensemble for $N=10$ at different γ . Here the solid line represents the density for $\gamma = 0$, the dashed line corresponds to $\gamma = 0.04$, the dashed-dotted line to $\gamma = 0.1$ and the dotted line to $\gamma = 0.69$.

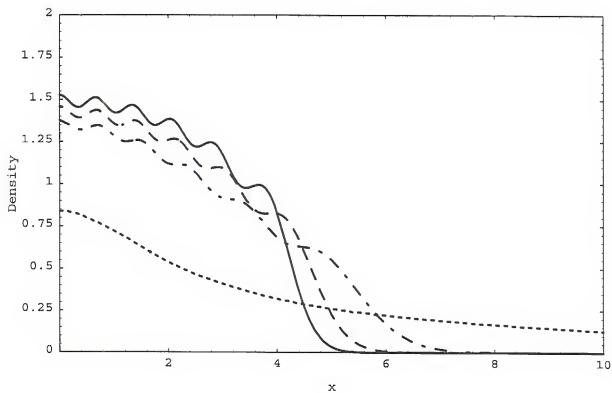


Figure 3.2: Eigenvalue density for the scaled q -Hermite model for $N=10$ at different γ . Here the solid line represents the density for $\gamma = 0$, the dashed line corresponds to $\gamma = 0.04$, the dashed-dotted line to $\gamma = 0.1$ and the dotted line to $\gamma = 0.69$

repulsion is considerably higher and therefore the eigenvalues are much further apart. Hence the level density is much smaller at the origin. This implies, as the integral over the density is constant, that the dropoff of the curve is less steep.

The density at the origin, as it is fairly constant, provides thus a measure for the mean level spacing of the ensemble and further rescaling. It can be determined with the help of the Christoffel-Darboux formula applied to q-Hermite polynomials.

$$\sqrt{w(x)w(y)} \sum_{n=1}^N H_n(x; q) H_n(y; q) = \frac{1}{k_n} \frac{h_{n+1}(x; q)h_n(y; q) - h_n(x; q)h_{n+1}(y; q)}{x - y} \quad (3.25)$$

by taking the limit $x \rightarrow y$. Here the H_n represent the normalized q-Hermite polynomials. Recognizing that odd and even q-Hermite polynomials have a different behavior at the origin we define

$$A_n = \lim_{x \rightarrow 0} h_{2n}(x; q) \quad (3.26)$$

and

$$B_n = \lim_{x \rightarrow 0} \frac{h_{2n+1}(x; q)}{x} \quad (3.27)$$

Calculating A_n

$$A_n = \frac{1-q}{q} \frac{1-q^3}{q^3} \cdot \frac{1-q^{2n-1}}{q^{2n-1}} \quad (3.28)$$

and similarly B_n

$$h_{2n+1}(x \rightarrow 0; q) = 2x h_{2n}(x; q) - \frac{1-q^{2n}}{q^{2n}} h_{2n-1}(x; q) \quad (3.29)$$

$$= -2x \frac{1-q^{2n-1}}{q^{2n-1}} h_{2n-2}(0; q) - 2x \frac{1-q^{2n}}{q^{2n}} h_{2n-2}(0; q) \quad (3.30)$$

$$+ \frac{1-q^{2n-2}}{q^{2n-2}} \frac{1-q^{2n}}{q^{2n}} h_{2n-3}(0; q) \quad (3.31)$$

$$= \dots \quad (3.32)$$

$$= -2x \left(\frac{1-q^{2n-1}}{q^{2n-1}} \frac{1-q^{2n-3}}{q^{2n-3}} + \frac{1-q^{2n}}{q^{2n}} \frac{1-q^{2n-3}}{q^{2n-3}} \cdot \frac{1-q}{q} \right) \quad (3.33)$$

$$+ \dots + \frac{1-q^{2n}}{q^{2n}} \frac{1-q^{2n-2}}{q^{2n-2}} \cdot \frac{1-q^4}{q^4} \frac{1-q^2}{q^2} \quad (3.34)$$

$$= -2x A_n \quad (3.35)$$

we get

$$\sigma_{2N}(0; q) = \frac{1}{\gamma(q; q)_\infty q^{-n(n-1)/2} (q; q)_\infty} A_n B_n \quad (3.36)$$

$$= \frac{1}{\gamma(q; q)_\infty} \sum_{k=1}^n q^k \prod_{j=1}^k \frac{1 - q^{2j-1}}{1 - q^{2j}} \quad (3.37)$$

$$= \frac{1}{\gamma(q; q)_\infty} \sum_{k=1}^n \exp(-\gamma k) \prod_{j=1}^k \frac{1 - \exp(-\gamma(2j-1))}{1 - \exp(-\gamma(2j))} \quad (3.38)$$

This expression can be well approximated by

$$\sigma_{2N}(0; q) \approx \begin{cases} \frac{\sqrt{N}}{\sqrt{\gamma}} & \gamma N < 1 \\ \frac{1}{\gamma} & \gamma N > 1 \end{cases} \quad (3.39)$$

A smooth approximation is given by

$$\sigma_{2N}(0; q) \approx \frac{1}{\sqrt{\gamma}} \frac{1}{\sqrt{N_q}} \quad (3.40)$$

where $N_q = \frac{1-q^N}{1-q}$. For $\gamma N \ll 1$ the expression 3.40 goes to the known Wigner semicircle law (except for the scaling parameter $\frac{1}{\sqrt{\gamma}}$). Also the limit $\gamma N \gg 1$ is recovered as required by equation 3.20. The density will be important for understanding the scaling properties discussed later.

3.2.2 The Kernel

In the method of orthogonal polynomials two level properties as the number variance and the Δ_3 -statistic are usually calculated with the help of the kernel. The kernel depends in a simple form on the eigenfunctions 3.12.

$$K(x, y; q) = \sum_{n=1}^N \phi_n(x; q) \phi_n(y; q) \quad (3.41)$$

The kernel for the q-Hermite polynomials can be exactly calculated for $N = \infty$, which was first shown in Ismail [76]. For this calculation one uses the following q-generalized linearization formula usually used to calculate a product of two polynomials as it appears in the kernel.

$$q^{m(m-1)/2} \frac{h_m(x; q)}{(q; q)_m} q^{n(n-1)/2} \frac{h_n(x; q)}{(q; q)_n} = \quad (3.42)$$

$$\sum_{k=0}^{\min(n,m)} \frac{q^{-n-m+k(k+1)/2+(m-k+1)(m-k)/2+(n-k+1)(n-k)/2}}{(q; q)_k (q; q)_{m-k} (q; q)_{n-k}} h_{m+n-2k}(x; q) \quad (3.43)$$

which expresses the product of two q -Hermite polynomials as a linear combination of q -Hermite polynomials. This linear combination is a generalization of the generating function

$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/2}}{(q; q)_{\infty}} h_n(x; q) = (-t(x + \sqrt{x^2 + 1}); t(\sqrt{x^2 + 1} - x; q)_{\infty} \quad (3.44)$$

which leads to (the details are provided in [91])

$$\sum_{n=0}^{\infty} h_n(\sinh(x); q) h_n(\sinh(y); q) \frac{q^{n(n-1)/2}}{(q; q)_{\infty}} R^n \quad (3.45)$$

$$= (-qRe^{x+y}, -qRe^{-x-y}, qRe^{x-y}, qRe^{y-x}; q)_{\infty} / (qR^2; q)_{\infty} \quad (3.46)$$

This formula gives the kernel (take $R=1$) for $N = \infty$

$$K(x, y; q) = \frac{w(x)}{\gamma(q; q)_{\infty}} \sum_{n=0}^{\infty} h_n(x; q) h_n(y; q) \quad (3.47)$$

$$= \frac{\sqrt{w(x)w(y)}}{\gamma[(q; q)_{\infty}]^2} (-qe^{x+y}, -qe^{-x-y}, qe^{x-y}, qe^{-x-y}; q)_{\infty} \quad (3.48)$$

The infinite products can be further expressed in Jacobi theta functions. Using the following relations

$$(qe^{x-y}, qe^{-(x-y)}; q)_{\infty} = \prod_{n=1}^{\infty} (1 - 2q^n \cosh(x-y) + q^{2n}) \quad (3.49)$$

$$= \frac{1}{2q^{1/8}(q; q)_{\infty}} \frac{\theta_1(i(x-y)/2; \sqrt{q})}{i \sinh((x-y)/2)} \quad (3.50)$$

and

$$(-qe^{x+y}, -qe^{-(x+y)}; q)_{\infty} = \prod_{n=1}^{\infty} (1 + 2q^n \cosh(x+y) + q^{2n}) \quad (3.51)$$

$$= \frac{1}{2q^{1/8}(q; q)_{\infty}} \frac{\theta_2(i(x+y)/2; \sqrt{q})}{i \cosh((x+y)/2)} \quad (3.52)$$

the kernel can be expressed as

$$K(x, y; q) = \frac{(\pi/\gamma)^{1/2}}{2 \ln(1/q) [(q; q)_{\infty}]^3} \frac{\cosh^{1/2}(x) \cosh^{1/2}(y)}{\cosh((x+y)/2)} \quad (3.53)$$

$$\frac{\theta_2(i(x+y)/2; \sqrt{q})}{\theta_2^{1/2}(ix; \sqrt{q}) \theta_2^{1/2}(iy; \sqrt{q})} \frac{\theta_1(i(x-y)/2; \sqrt{q})}{i \sinh((x-y)/2)} \quad (3.54)$$

Since the imaginary arguments are difficult to evaluate, Jacobi's imaginary transformations are applied

$$\theta_2(ix; \sqrt{q}) = (\pi/\gamma)^{1/2} e^{x^2/\gamma} \theta_4(\pi x/\gamma, e^{-2\pi^2/\gamma}) \quad (3.55)$$

$$\theta_1(ix; \sqrt{q}) = i(\pi/\gamma)^{1/2} e^{x^2/\gamma} \theta_1(\pi x/\gamma; e^{-2\pi^2/\gamma}) \quad (3.56)$$

resulting in

$$K(x, y; q) = \frac{(\pi/\gamma)^{1/2} \cosh^{1/2}(x) \cosh^{1/2}(y)}{2 \ln(1/q) [(q; q)_\infty]^3} \frac{\cosh((x+y)/2)}{\cosh(\pi(x+y)/2\gamma; e^{-2\pi^2/\gamma})} \frac{\theta_1(\pi(x-y)/2\gamma; e^{-2\pi^2/\gamma})}{\theta_4^{1/2}(\pi x/\gamma; e^{-2\pi^2/\gamma}) \theta_4^{1/2}(\pi y/\gamma; e^{-2\pi^2/\gamma}) \sinh((x-y)/2)} \quad (3.57)$$

In the limit $\gamma \ll 2\pi^2$, $y \approx x$ one can further simplify the result by approximating the theta functions by

$$\theta_1(z, q) = 2q^{1/4} \sin(z) \quad (3.58)$$

$$\theta_4(z, q) = 1 \quad (3.59)$$

finally obtaining (and using $\frac{1}{[(q; q)_\infty]^3} = \exp(\pi^2/4\gamma)(\frac{\gamma}{\pi})^{3/2}$)

$$K(x, y; q) = \frac{1}{2\pi} \frac{\cosh^{1/2}(x) \cosh^{1/2}(y)}{\cosh((x+y)/2)} \frac{\sin(\pi(x-y)/2\gamma)}{\sinh((x-y)/2)} \quad (3.60)$$

This formula represents a fundamental result for the q-Hermite model [77]. In order to concentrate on the bulk behavior the assumption $x - y \ll 1, x + y \gg 1$ can be imposed, which simplifies the kernel to

$$K(x, y; q) = \frac{1}{2\pi} \frac{\sin[\pi(x-y)/2\gamma]}{\sinh((x-y)/2)} \quad (3.61)$$

This kernel has apparently the density

$$K(x, x; q) = \sigma(x; q) = \frac{1}{2\gamma} \quad (3.62)$$

which is constant but not unity. Fortunately, the transformation which leads to unit density and therefore an unfolded spectrum is obvious. With $x \rightarrow x/\gamma, y \rightarrow y/\gamma$ we arrive then at the scaled bulk kernel

$$\bar{K}(x, y; q) = \frac{1}{2\pi} \frac{\sin(\pi(x-y))}{\sinh(\gamma(x-y)/2)} \quad (3.63)$$

In order to interpret this result the reduced two level probability function $R_2 = 1 - \overline{K}^2$ as a function of the distance $z = x - y$ is plotted for different γ in figure 3.3. Note that only the square of the kernel has a probability meaning and is hence physically relevant.

The resulting limiting cases are easily understood by noting that in the Wigner-Dyson limit the level repulsion prohibits levels close to each other. Accordingly, the two level probability function is zero when the distance between the two levels is negligible. When the levels are far apart they don't influence each other, and the two level probability function is nonzero (scaled to one in the plot). On the other hand, if the eigenvalues behave in accordance to Poisson-statistics, they are entirely independent from each other, leading to a constant two level probability function even at small argument.

These limiting cases are well known, however, the qualitative form of the transition between Wigner-Dyson and Poisson-statistics is characteristic for the q-Hermite model and can be compared with either experimental data or numerical experiments. The power law decay of the Wigner-Dyson statistics (obvious in the Kernel $K_{W-D}(x, y) = \frac{1}{4\pi^2} \frac{\sin^2(\pi(x-y))}{(\pi x)^2}$) is replaced by the exponential decay of the q-Kernel(). Note that the exponential decay is present for all $\gamma > 0$, indicating a fundamental difference between the Wigner-Dyson limit and the extension to $q < 1$. This behavior represents one example, where it becomes visible that the extended (q-deformed) random matrix models belong to a different universality class of random matrix theories compared to 'classical' random matrix theories. In fact, it can be expected that all q-deformed models give the same universal behavior, thereby indicating the same universality class, because their one body potential has always a lognormal behavior for large eigenvalues. This assertion can be supported by investigating another q-deformed ensemble, and showing that the kernel is quantitatively the same in the bulk. This has explicitly been done for the q-Laguerre ensemble [45, 46].

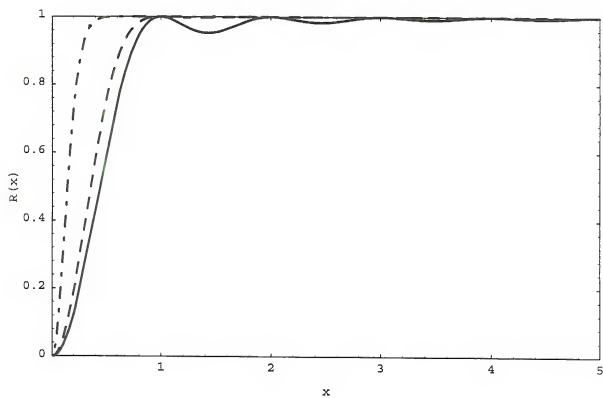


Figure 3.3: Reduced two level probability function for $N = \infty$ and different γ . Here the solid line represents $\gamma = 0$, the dashed line represents $\gamma = 5$ and the dashed-dotted line represents $\gamma = 20$.

Because of the importance of this argument and the brevity of the calculation in the reference mentioned, we will outline the calculation here in more detail. The q -Laguerre ensemble is defined by the following power series:

$$L_n(x; q) = \sum_{k=0}^n \frac{(q^{-n}; q)_k q^{k(k-1)/2} (1-q)^k (q^{n+1}x)^k}{(q; q)_k^2} \quad (3.64)$$

This definition requires, as before for the q -Hermite polynomials, the knowledge of the recursion relation as well as the normalization. Additionally, the invariant measure is required.

$$w_L(x; q) = \frac{1}{(-(1-q)x; q)_\infty} \quad (3.65)$$

It was shown [46], that this invariant measure fulfills the requirements for a q -generalized model (strong confinement at the origin and weak confinement at large coordinate). The Christoffel-Darboux formula [4] for the q -Laguerre ensemble is used for calculating the kernel

$$K_{L,N}(x, y; q) = \sqrt{w_L(x; q)w_L(y; q)} \sum_{n=0}^N \frac{1}{k_n} L_n(x; q) L_n(y; q) \quad (3.66)$$

where $k_n = \frac{\gamma}{(1-q)q^n}$. Applying the formula one obtains

$$K_{L,N}(x, y; q) = \frac{1 - q(q; q)_{N+1}}{\gamma (q; q)_N} \frac{1}{(1-q)q^{N+1}} \sqrt{w_L(x; q)w_L(y; q)} \quad (3.67)$$

$$\frac{L_{N+1}(y; q)L_N(x; q) - L_N(y; q)L_{N+1}(x; q)}{x - y} \quad (3.68)$$

$$\stackrel{=N \rightarrow \infty}{=} \frac{1 - q^{N+1}}{\gamma} \sqrt{w_L(x; q)w_L(y; q)} \quad (3.69)$$

$$\frac{L_\infty(y/q; q)L_\infty(x; q) - L_\infty(y; q)L_\infty(x/q; q)}{x - y} \quad (3.70)$$

The occurrence of the function L_∞ reflects the predictability of the q -deformed Laguerre polynomials at large N . The function L_∞ can be approximated for $(1-q)x \gg 1$ by

$$L_\infty(x; q) = J_0^{(2)}(2\sqrt{(1-q)x}; q) \quad (3.71)$$

$$\approx \frac{4\pi}{\gamma} \cos\left(\frac{\pi}{\gamma} \ln(\sqrt{(1-q)x})\right) e^{\frac{1}{\gamma}(\ln^2(\sqrt{(1-q)x}) - \frac{\pi^2}{4})} \quad (3.72)$$

The potential $w_L(x; q)$ is expanded by use of Jacobi's triple product formula:

$$(q; q)_\infty (-zq^{1/2}; q)_\infty \left(\frac{q^{1/2}}{-z}; q\right) = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n \quad (3.73)$$

Choosing $z = \frac{(1-q)x}{\sqrt{q}}$ and realizing that the first and last product are of order one, one gets

$$(-(1-q)x; q)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2/2} \left(\frac{(1-q)x}{\sqrt{q}}\right)^n \quad (3.74)$$

Approximating the sum by an integral the potential term becomes

$$\sqrt{w_L(x; q)} = \left(\frac{2\pi}{\gamma}\right)^{1/4} \exp\left(-\frac{1}{4\gamma} [\ln^2((1-q)x) + \gamma/2]^2\right) \quad (3.75)$$

After inserting these relations into the kernel one obtains

$$\begin{aligned} K_L(x, y; q) &= \sqrt{\frac{2\pi}{\gamma}} \frac{1 - q^{N+1}}{\gamma} \left[\left(\frac{x}{y}\right)^{1/4} \cos\left(\frac{\pi}{2\gamma} \ln((1-q)y)\right) \sin\left(\frac{\pi}{2\gamma} \ln((1-q)x)\right)\right. \\ &\quad \left.- \left(\frac{y}{x}\right)^{1/4} \sin\left(\frac{\pi}{2\gamma} \ln((1-q)y)\right) \cos\left(\frac{\pi}{2\gamma} \ln((1-q)x)\right)\right] \end{aligned}$$

Applying the similarity transformation

$$\xi = \frac{1}{2} \ln((1-q)x) \quad (3.76)$$

$$\eta = \frac{1}{2} \ln((1-q)y) \quad (3.77)$$

the integration element $w(x)dx$ has to be kept constant, i.e. $w(x)dx = \overline{w}(\xi)d\xi$. The result is then

$$K(\xi, \eta; q) = \frac{1}{2\pi} \frac{1}{e^\xi + e^\eta} \left(\frac{e^\xi \cos(\frac{\pi}{\gamma}\eta) \sin(\frac{\pi}{\gamma}\xi) - e^\eta \sin(\frac{\pi}{\gamma}\xi) \cos(\frac{\pi}{\gamma}\eta)}{\sinh(\frac{\xi-\eta}{2})} \right) \quad (3.78)$$

In the bulk, when $\eta, \xi \ll 1$, $\eta \approx \xi$, the exponential functions will vanish leading then to the q-Hermite kernel 3.61 This proves the assumption that in the bulk the q-Hermite and the q-Laguerre ensemble have the same behavior. This result also strongly suggests that every q-deformed model will have the same and therefore universal behavior in the bulk. Only close to the origin the ensembles will have a markedly different behavior. As expected, this property has an analogue for Wigner-Dyson ensembles.

3.2.3 The Two-Level Form Factor

Based on the kernel calculated before, physically relevant quantities can be calculated. The two-level form factor, which is simply the Fourier transform of the two-level Cluster function, is one important example. It reads

$$b(k) = \int_{-\infty}^{\infty} e^{ik(x-x')} Y(x-x') d(x-x') \quad (3.79)$$

$$= \int_{-\infty}^{\infty} e^{ik(x-x')} K^2(x-x') d(x-x') \quad (3.80)$$

$$= \frac{\gamma^2}{4\pi^2} \int_{-\infty}^{\infty} \frac{\sin(\pi(x-x'))}{\sinh(\frac{\gamma}{2}(x-x'))} e^{ik(x-x')} d(x-x') \quad (3.81)$$

This integral can be calculated exactly (compare with Appendix B) with the result

$$b(k) = \frac{1}{4\pi} ((|k|+2\pi)\coth[(|k|+2\pi)\pi/\gamma] + (|k|-2\pi)\coth[(|k|-2\pi)\pi/\gamma] - 2|k|\coth(|k|\pi/\gamma)) \quad (3.82)$$

By expanding the cotangent hyperbolic in the limit $\gamma \rightarrow 0$, the known Wigner-Dyson limit is recovered

$$b(k) = \begin{cases} 1 - \frac{|k|}{2\pi} & |k| < \pi \\ 0 & |k| > \pi \end{cases} \quad (3.84)$$

At $k=0$, which is just the integral over the Cluster function, the two level form factor takes the value

$$b(0) = \coth\left(\frac{2\pi^2}{\gamma}\right) - \frac{\gamma}{2\pi^2} \quad (3.85)$$

Note the nonanalytic dependence on γ , which excludes the possibility to obtain the above result by a perturbative approach from the Wigner-Dyson limit.

The Fourier transform of the two point correlation function or two-level form factor has recently been of interest for experimentalists and theorists alike [80, 78, 79]. The properties of a spectrum are quite difficult to measure experimentally due to its highly oscillating nature. However, the Fourier transform can reveal 'used' or 'unused' frequencies and should therefore represent a more robust measure of the eigenvalue

correlations. Indeed, in the case of random matrices where the level repulsion is the fundamental phenomenon, the Fourier transformation should lead to the occurrence of a correlation hole. The two-level form factor and the occurring correlation hole are displayed in figure 3.4. Smaller frequencies in the spectrum are suppressed, because small energy differences are statistically disfavored.

3.3 Statistical Analysis

3.3.1 Nearest Neighbour Spacing Distribution

An important quantity for transition models is the nearest neighbour spacing distribution. The spacing distribution cannot be accessed with the correlation functions alone, but additionally requires to solve an integral equation [4]. This difficulty makes it appropriate to calculate the spacing distribution numerically. Figure 3.5 shows a clear discrepancy between the Wigner-Dyson statistics and Poisson statistics and is fairly easy to evaluate for numerical eigenvalue distributions. For the transition model it is particularly interesting to see how the Wigner 'surmise' crosses over into the Poisson neighbour distribution. This family of curves shows a remarkable similarity with numerous investigations of transitional statistics from many sources (as a few examples [8, 67, 48]). One interesting feature visible from the nearest neighbour spacing distribution is the still active level repulsion even at finite γ . Due to the standard logarithmic repulsion term the curves start always at zero. However, the slope at the origin clearly increases, which shows the qualitatively different nature of the q -Hermite ensemble, whereas the slope is constant for fairly general circumstances (see equation 1.32) in standard random matrix theory in the same symmetry class. The maximum of the curve slowly moves towards the origin, as widely observed [8, 81, 36]. This feature is not reproduced in other purely phenomenological or semiphenomenological theories [15, 16].

Of even more speculation is the obvious 'scaling' point, where all curves intercept. The long range decay of the nearest neighbour spacing distribution is in case of a

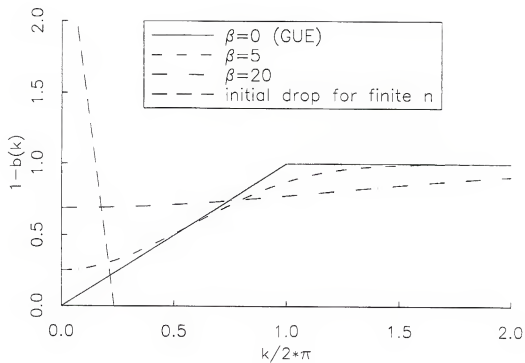


Figure 3.4: Two level form factor for $N = \infty$ at different γ . The minima at small energies can be interpreted as a correlation hole. The figure was taken from reference [52] where the parameter β is identical to the parameter γ used in here.

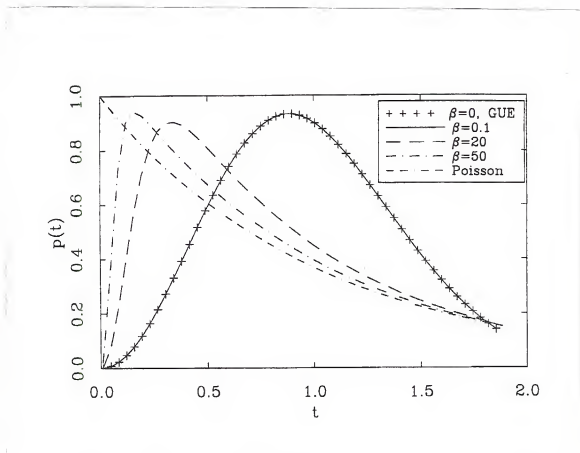


Figure 3.5: Nearest neighbour spacing distribution for $N = \infty$ at different γ . The variable t is given by $t = x/D$, where x is the coordinate and D the mean level spacing. The figure was taken from reference [77] where the parameter β is identical to the parameter γ used in here.

Poisson distribution apparently $\exp(-x)$ in contrast to the $\exp(-x^2)$ decay in the Wigner-Dyson case. This large difference in the decay behavior has prompted additional interest, because it could lead to the actual 'sharp' transition features, which are so far hidden. One idea is to define the integral from the scaling point to infinity (the tail of the distribution) as the order parameter of the transition [53]. Another approach is to calculate the tail of the distribution for the q-Hermite model in a more analytical fashion [82].

3.3.2 Number Variance and Δ_3 -statistics

In contrast to the nearest neighbour spacing distribution, which manifestly measures the eigenvalue correlations for close range eigenvalues, the number variance and the Δ_3 -statistics are long range measures of the eigenvalue correlations. This is achieved by integrating a correlation function over the entire spectrum. Therefore it includes also the effect of eigenvalues which are far apart. The number variance V_N in the interval $(-N/2, N/2)$ is given by [4]

$$V_n(N) = \overline{n^2} - \bar{n}^2 \quad (3.86)$$

$$= \int_{-N/2}^{N/2} dx \int_{-N/2}^{N/2} dy [\delta(x-y) - Y(x-y)] \quad (3.87)$$

$$= \frac{2}{\pi} \int_0^\infty dk \frac{1 - \cos(kN)}{k^2} [1 - b(k)] \quad (3.88)$$

with the two level form factor $b(k)$ covered before. To an excellent approximation, $b(k)$ contributes negligibly to the integral in equation 3.86 for $k > 2\pi$. We find [52]

$$V_n(N) = [1 - \rho]N + \frac{\rho}{\pi^2} [Ci(2\pi s) + 1] \quad (3.89)$$

$$+ \frac{1}{\pi^2} \int_0^{2\pi N} dk \frac{1 - \cos(kN)}{k} \left[\coth\left(\frac{k\pi}{\gamma N}\right) - \frac{\gamma s}{k\pi} \right] \quad (3.90)$$

where Ci is the cosine integral, and ρ defined as

$$\rho = \coth\left(\frac{2\pi^2}{\gamma}\right) - \frac{\gamma}{2\pi^2} \quad (3.91)$$

goes to 1(0) for $\gamma \rightarrow 0(\infty)$. We can obtain an explicit expression for the number variance by approximating the factor $[\coth(\frac{k\pi}{\gamma N}) - \frac{\gamma N}{k\pi}]$ under the integral by $k\pi/3\gamma N$ up to $k = 3\gamma N\rho/\pi$ and by ρ for larger k . The result

$$V_n(N) = [1 - \rho]N + \frac{\rho}{\pi^2} \left[1 - \frac{\sin(3\gamma\rho N/\pi)}{3\gamma\rho N/\pi} \right] \quad (3.92)$$

$$+ \frac{\rho}{\pi^2} \left[\ln \left(\frac{2\pi^2}{3\beta\rho} \right) + Ci \left(\frac{3\gamma\rho N}{\pi} \right) + 1 \right] \quad (3.93)$$

agrees with the numerical evaluation of the integral up to a small γ dependent constant for large N as well as corrections of order $1/N$. Equation 3.92 clearly exhibits the crucial dependence of the number variance on the parameter γ . As expected, $\gamma = 0$ corresponds to the exact GUE result, with a logarithmic dependence on N ; increasing γ corresponds to a transition towards a Poisson result, which is linear in N . Note that the variance of an arbitrary linear statistic $f = \sum_n f(x_n)$, of which V_n is a special case, will depend on the form factor 3.79 and is clearly no longer universal

$$\text{Var}(f) = \frac{2}{\pi} \int_0^\infty dk [1 - b(k)] |\bar{f}(k)|^2 \quad (3.94)$$

where $\bar{f}(k)$ is the Fourier transform of $f(x)$. This result generalizes the Dyson-Mehta theorem[7] on the variance of arbitrary linear statistics and reduces to it when $\gamma \rightarrow 0$. The Δ_3 -statistics is a quite popular measure of long range eigenvalue correlations. Already introduced in the first chapter, the Δ_3 -statistics is a good measure of the rigidity of a given eigenvalue spectrum. It exemplifies the large difference between a Poisson spectrum and Wigner-Dyson spectrum by showing a linear and therefore large deviation from a perfectly rigid spectrum in the Poisson case and a logarithmic and much more weaker deviation from the Wigner-Dyson case. This qualitatively different behavior motivates the calculation of the Δ_3 -statistics for the transition model. The q-Hermite model cannot be evaluated in closed form but the interesting limiting cases are analytically accessible. The Δ_3 -statistics can be written in the following form:

$$\Delta_3 = \frac{N}{15} - \frac{1}{15N^4} \int_0^N du \frac{\sin^2(\pi u)}{\sinh^2(\gamma u/2)} [3u^5 - 20u^3N^2 + 30u^2N^3 - 15uN^4 + 2N^5] \quad (3.95)$$

For small γ (close to Wigner-Dyson) we can calculate the first γ dependent correction, by expanding the kernel $\frac{\gamma^2}{4\pi^2} \frac{\sin^2(\frac{\pi x}{\gamma})}{\sinh^2(\frac{1}{2}\gamma x)}$ to first order in γ . The result is then [52]

$$\Delta_3(N, \gamma) = \frac{1}{2\pi} \left[\ln(2\pi s) + \gamma - \frac{5}{4} \right] + \frac{\gamma^2 N^2}{72\pi^2} + O((\gamma N)^4) \quad (3.96)$$

for $\gamma \ll 1$. This result reduces obviously to the Wigner-Dyson limit ($\gamma = 0$) for $\gamma \rightarrow 0$. On the other hand for large γ the large N limit is appropriate, where the finite integral can be expanded around the $N = \infty$ -case.

$$\int_0^N = \int_0^\infty - \int_N^\infty \quad (3.97)$$

Then we find

$$\frac{\Delta_3(N, \gamma)}{s} = \frac{1}{15} \left[\frac{\gamma}{2\pi^2} - \frac{2}{\exp\left(\frac{4\pi^2}{\gamma}\right) - 1} \right] + C + O(e^{-\gamma s}) \quad (3.98)$$

for $\gamma \gg 1/N$, which reduces to the Poisson result for $\gamma \rightarrow \infty$. Here C is a numerical constant independent of N and very weakly dependent on γ . Note that since the ratio Δ_3/N is finite, Δ_3 always has a linear dependence for large enough N , with a slope that increases with increasing γ , approaching the Poisson limit for $\gamma \rightarrow \infty$. For values in between a numerical evaluation is necessary. For a small set of eigenvalues (N small) the family of curves shows still a logarithmic increase of the Δ_3 -statistics. However, for larger N the curve smoothly crosses over into a linear increase. The slope of the large N , linear increase is uniquely dependent on γ and thus indicates how far the statistical transition has progressed. Figure 3.6 shows the complete solution (obtained from numerical evaluation of the integrals in equation 3.95) for various values of γ . The deviation from the GUE result as a function of some parameter is qualitatively similar to the deviations seen numerically in both the transfer matrix [50] and energy eigenvalues [48] corresponding to the tight binding Anderson Hamiltonian for disordered conductors, as well as for the eigenvalues of the evolution operator corresponding to the Fermi-acceleration model [83].

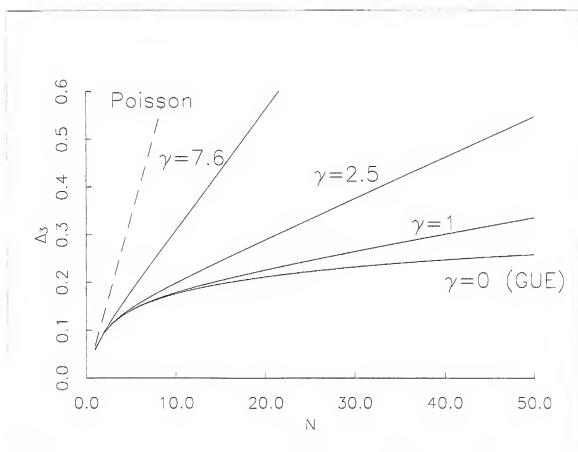


Figure 3.6: Δ_3 -statistics for $N = \infty$ at different γ .

Also note that the slope exhibits a manifestly nonanalytic dependence on γ , showing the nonperturbative nature of the transition model. This also indicates the subtle nature of the behavior from $\gamma = 0$ to $\gamma \rightarrow 0$ in the present formulation. A closer analysis reveals that this behavior comes from the fact that the results are all evaluated for an infinite set of eigenvalues. Arguments presented later in this chapter as well as in the dynamic model formulation show that the limit $\gamma \rightarrow 0$ is only properly understood for a finite number of eigenvalues. This originates from the fact that the limits $\gamma \rightarrow 0$ and $N \rightarrow \infty$ do not interchange.

The behavior of the number variance and the Δ_3 -statistics are apparently quite similar, even though they contain slightly different information. The reason for this similarity is the analytical agreement at large- N , where the Δ_3 -statistics retains only the first integral in 3.95 as a correction to the linear term, i.e.

$$\Delta_3(s \ll 1) = \frac{1}{15}s - \frac{2s}{15} \frac{\gamma^2}{4\pi} \int_0^s du \frac{\sin^2(\pi u)}{\sinh^2(\gamma u/2)} \quad (3.99)$$

which is precisely, apart from constants, the terms contained in the number variance 3.92. Consequently, the large N behavior is indeed similar. However, the small N characteristics are certainly different as now all integral terms in expression 3.95 contribute to the Δ_3 -statistics.

The small N behavior of the Δ_3 -statistics is relevant for some models. Numerical investigations with the tight binding Anderson Hamiltonian [48] have shown that for small N the Δ_3 curve can be fitted with a power law $\Delta_3(N) \propto N^\alpha$, where the coefficient α is a characteristic of the physical system in question. But the transition model discussed herein is only able to describe the gross features of a transitional system and is therefore not able to recover the model dependent features at small N . A simple fitting procedure for the coefficient gives an alpha of 0.63 for the transitional model, comparing well with the alpha of 0.70 for the analysis of the eigenvalue statistics of the tight binding Andersson model.

To better understand the range of the transition, one can plot the Δ_3 -statistics

as a function of γ at a fixed but sufficiently large N . The result is depicted in figure 3.7 where the ordinate is normalized by the leading linear term. Apparently the transition is very much linearly dependent on the parameter γ . The saturation at larger γ is not relevant, as the kernel 3.61 used in this calculation is no longer valid.

3.4 Long-Range Off-Diagonal Oscillations

So far the investigations of the kernel have concentrated on the bulk behavior, which is close to the main diagonal ($x \approx y$) and far away from the origin, which is certainly very important for the characterization of q -models. The complete similarity in the kernels at the bulk for both q -Hermite and q -Laguerre models (cmp. eq()), supports this statement. However, close to the origin the q -models have a different behavior and show therefore a system dependent behavior. This model dependency can be used to construct a specific theory, the most well known case is the transfer matrix formulation of conductance ([21] or for references see [85]), where the correlation function values close to the origin determine measurable quantities like the conductance.

The q -Hermite model behaviour at the origin is more difficult to connect with physically measurable quantities, because the eigenvalue spectrum does not lead straightforward to a physical quantity. However, the inclusion of an external parameter into the q -Hermite model gives again reason to raise the question of the behavior close to the origin and is discussed in the next chapter.

In the case of the q -Hermite model another phenomenon occurs which is based on the reflection symmetry towards the origin of the underlying polynomials. The two point correlation function shows long range oscillations in the off-diagonal direction (i.e. the $(x, -x)$ -direction on a x - y -plane). These oscillations do not occur, if the system is in the Wigner-Dyson limit, because the correlation function depends in this case only on the energy difference variable. This translational symmetry is spontaneously

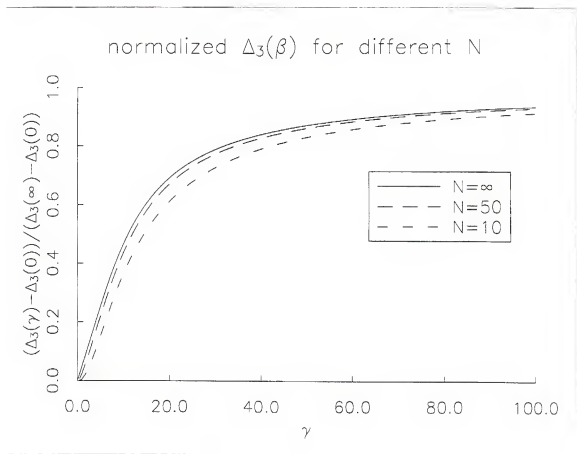


Figure 3.7: Normalized Δ_3 -statistics as function of γ for $N = \infty$.

broken in the case that the disorder parameter γ is nonzero. It has to be emphasized that the long range oscillations will only appear in the q-Hermite model, and are therefore model dependent.

With the help of the kernel we can characterize these oscillations. Using the equation 3.60 with the scaled coordinates ($x \rightarrow \gamma x$, $y \rightarrow \gamma y$) yields

$$K(x, y; q) = \frac{\gamma}{2\pi} \frac{\sqrt{\cosh(\gamma x) \cosh(\gamma y)}}{\cosh(\gamma(x+y)/2)} \frac{\sin(\pi(x-y))}{\sinh(\gamma(x-y)/2)} \quad (3.100)$$

Concentrating on the off-diagonal direction $y = -x$ the result is

$$K(x, -x, q) = \frac{\gamma}{2\pi} \coth(\gamma x) \sin(2\pi x) \quad (3.101)$$

The cotangent hyperbolic factor will approach one for large enough x . Then it is easy to see that the long range oscillations have a frequency of 2π and grow linearly with γ as long as $\gamma \ll 2\pi^2$. The whole kernel, based on equation 3.47 is plotted in figure 3.8, clearly exhibiting the oscillations also for $\gamma \approx 1$.

Note that the value for γ is already relatively large, mainly to illustrate the off-diagonal oscillations. A comparison with the analytical expression 3.101 would predict that the height of the peak is close to $(\frac{\gamma}{2\pi})^2 \approx 0.5$ which is a small underestimation of the peak height from the plot (about 0.4). This shows already that the regime, where formula 3.101 is valid, is starting to give imprecise results.

It is instructive to go to even larger γ and plot the full kernel again. Figure 3.9 displays the kernel for $\gamma = 8.1$. The off-diagonal oscillations are at this value at full height and saturated. Interesting, however, is the formation of similar oscillation peak on the main diagonal. This phenomenon is certainly not described by the kernel 3.61 as $\gamma > 1$, but can be understood by considering properties of the q-Hermite polynomials.

The zeroes of q deformed polynomials are much more likely to be found at one position, compared to the undeformed case, where the zeroes cannot be fixed with increasing N . This 'ergodic' behavior of the zeroes in the Wigner-Dyson case leads to

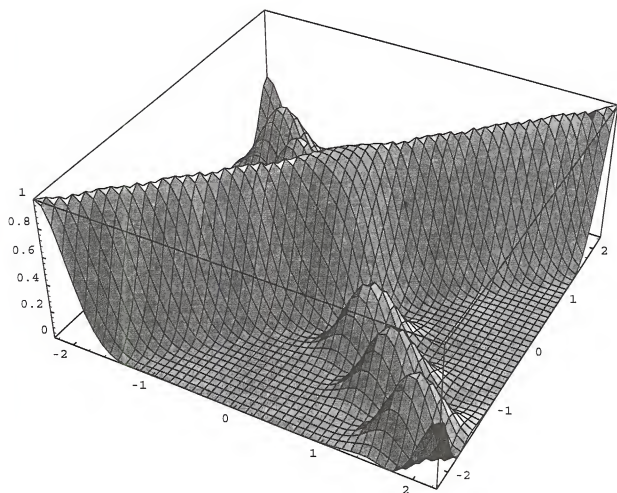


Figure 3.8: Full $N = \infty$ -kernel K^2 for $\gamma = 4.6$ for $N = \infty$.

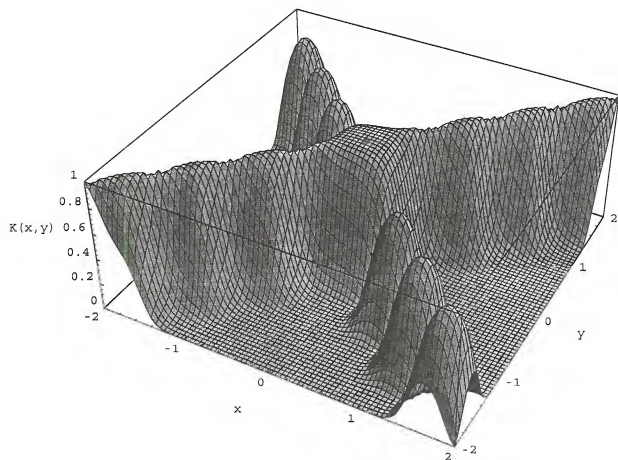


Figure 3.9: Full $N = \infty$ -kernel K^2 for $\gamma = 8.1$ for $N = \infty$.

a very smooth and featureless correlation functions, visible in translational invariance. However, for $q < 1$ the eigenvalues are more likely to stay at one place with increasing N , as the zeroes of the polynomials give the most likely position of the eigenvalues. This fundamental difference between the Wigner-Dyson case and the deformed (to be more precise any q deformed) ensemble is illustrated by the existence of a $N = \infty$ limiting function for the polynomials, as already used in equation 3.71 for the Laguerre ensemble.

The oscillations in the $(x, -x)$ -direction may have some physical relevance. Consider an energy spectrum of a physical system in transition between metallic and insulating behavior, which preferably exhibits a band structure. For comparing the results from random matrix theory with the energy eigenvalue spectrum one limits them to be from the 'bulk' (meaning far from the band edges). As long as the eigenvalues are not influenced by the band edge behavior, their correlations should be uniform. Therefore the system in the bulk exhibits clearly translational symmetry. The fact that in the case of the q -Hermite model the translational symmetry around a specific point (the origin) is broken could indicate the potential relevance of the q -Hermite model in understanding the effect of the band edge.

An important test for this assumption is to calculate an integral quantity such as the number variance, which uses an integration region including the origin. As the origin plays in the above speculation the role of the band edge, it is appropriate to integrate over a region symmetric towards the main diagonal, but far away from the origin for small N . Only at larger N the origin would be included, and the effect of the long range oscillations becomes visible. The result of this number variance calculation is shown in figure 3.10. As expected, at small N the origin is not included, and the number variance resembles the results from the known expression 3.92. However, when N is large, and the integration includes the origin, the behavior of the number variance shows leveling, constant number variance.

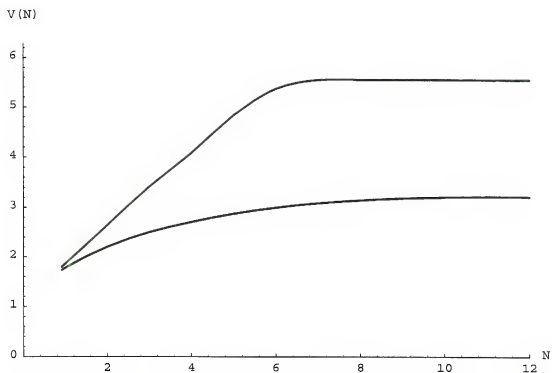


Figure 3.10: Number variance over the full kernel. The upper line represents $\gamma = 4.6$ and the lower line corresponds to $\gamma = 1.2$.

3.5 Finite-N Analysis

The discussion so far has focussed on the investigation for infinite many eigenvalues. The results show a qualitatively different behavior of the q-Hermite model, compared to the standard random matrix Hermite model. However, the crossover from one model to the other is difficult to apprehend for $N = \infty$ results. This problem is apparent in all the above mentioned results, which are based on the $(N = \infty)$ -kernel 3.61. The kernel already has a qualitatively different behavior, where the power law decay is replaced by an exponential one. This difference is even more pronounced in quantities which integrate over the spectrum, including eigenvalues far apart, such as the number variance and the Δ_3 -statistics. Figure 3.7 indicates the existence of a noncontinuous crossover from $\gamma \ll 1$ to $\gamma = 0$. No matter how small γ is, the integral measures will contain a linear term which will dominate for sufficiently large N . This behavior prohibits a smooth crossover.

In order to compare the finite N results with the $N = \infty$ calculations one has to normalize the correlation functions correctly. The principle relation to use here is of course the normalization of the joint probability distribution function $\int P(\{x_n\})d\{x_n\} = 1$. For finite N this leads to the earlier defined reduced probability distribution functions R_n (in particular $R_2(x, y) = \sigma(x)\sigma(y) - K^2(x, y)$) and

$$\int R_2(x, y)dx dy = N(N - 1) \quad (3.102)$$

one has

$$\int \sigma(x)dx = N \quad (3.103)$$

$$\int K^2(x, y)dx dy = N \quad (3.104)$$

To scale the energy scale out of the problem, i.e. the mean level spacing D , the energy levels are rescaled such that the level density is constant. In case one chooses

the constant to be unity the energy scale is no longer present in the problem. The rescaling is represented by a change of variables from $\{x_n\}$ to $\{x'_n\}$ and the correlation functions in the new variable have to satisfy the same integral constraints 3.103 as in the old variable set. For the density this leads to the following relation

$$\sigma_N(x)dx = \sigma_N(x')dx' = [\Theta(x' + N/2) - \Theta(x' - N/2)]dx' \quad (3.105)$$

where $\Theta(x) = 1$ for $x > 0$ and zero else. The last equation comes from the choice of unit density in the new variables. However, the choice of density $\sigma(x')$ is not unique. Similarly the two point correlation measure obeys

$$R_{2,N}(x, y)dx dy = R_{2,N}(x', y')dx'dy' \quad (3.106)$$

leading to

$$K_N^2(x, y)dx dy = K_N^2(x', y')dx'dy' \quad (3.107)$$

Using equation

$$K_N^2(x', y') = \frac{K_N^2(x, y)}{\frac{dx'}{dx} \frac{dy'}{dy}} \quad (3.108)$$

$$= \frac{K_N^2(x, y)}{\sigma(x)\sigma(y)} \quad (3.109)$$

Inserting the formulas for the orthogonal polynomial expressions 2.25, 2.26, one arrives at

$$K_N^2(x', y') = \frac{(\sum_{n=0}^{N-1} h_n(x; q)h_n(y; q))^2}{\sum_{n=0}^{N-1} h_n^2(x; q) \sum_{n=0}^{N-1} h_n^2(y; q)} \quad (3.110)$$

For all q-models the two point correlation function decays in the bulk ($x, y \gg 1$) as a characteristic for the class of q-models investigated.

To compare the kernel for finite N with the results already obtained one can focus on the bulk behavior. For this purpose we can compare the sum 3.110 for increasing N and compare with the $N = \infty$ kernel. One should concentrate on the scaled polynomials in 3.24, by comparing the function ($x' = \int^x \sigma(x)dx$, $K_N(x + N/2, -x + N/2)$)

and compare with the kernel 3.61. The shift in the kernel arguments is necessary, we would like to observe explicitly bulk behavior. The results have to be scaled to result in unit density.

This scaling becomes clear, once being reminded of the fact that the zeroes of the Hermite H_n polynomials scale with $\sqrt{n}$. Dividing the argument by this number results in a curve which can be compared to the analytical kernel. However, for stronger q deformation the movement of the zeroes saturates at $\frac{1}{\gamma}$. As also used later, a good intermediate approximation is $\sqrt{n_q}$.

This approach leads to a good description of the scaled q -Hermite polynomials and the kernel associated with them. However, the procedure for the unscaled polynomials results in an unsatisfactory description.

CHAPTER 4

THE DYNAMIC DEFORMED GAUSSIAN ENSEMBLE

The transitional model discussed in here can be extended to include an external perturbation. This perturbation is usually an external field acting upon the system. As this dependence allows to predict new phenomena and to explain perturbation-dependent data, the physical relevance is immediately obvious.

The parameter-dependent theory can be treated in an approximate and an exact fashion. The approximation provides easier access to some important results as well as better understanding of regions where the exact treatment is quite complicated to evaluate. In the following both approaches are discussed in detail, and the results are put into the context of parametric correlations for chaotic spectra.

4.1 The Hydrodynamic Approximation

4.1.1 Brownian Motion Model

The standard random matrix model describes a system of eigenvalues in static equilibrium. Because of the close analogy to a 2D electron gas the model was also called Coulomb gas model. Dyson introduced the idea to extend the modeling efforts to include a dynamic dependence on an external parameter. Dynamic means here dependence of the eigenvalues on some 'fictitious' time, which doesn't have to be the physical time but any other external parameter on which the eigenvalues depend. Dyson realized that a description in form of a conservative, newtonian formulation cannot be done in a reasonable physical way [6, 86], i.e. defining velocities and inertia of the eigenvalues (or particles in the Coulomb gas analogy) is impossible. However, a nonconservative, dissipative description does not require the definition of the velocity of a single eigenvalue. The most straightforward ansatz is thus to assume that the eigenvalues will

evolve according to Brownian motion. The underlying principle of the Brownian motion model are the frictional forces each particle is exposed to. The physical picture implies that the eigenvalues are experiencing local friction forces given by a usual second derivative of the distribution function and an external 'electric' field which is the derivative of the potential, i.e.

$$\mathcal{E}(\{E_n\}) = -\frac{\partial W}{\partial x_i} = \sum_{i, i \neq j} \left[\frac{1}{E_i - E_j} \right] - 2E_i \quad (4.1)$$

The fluctuating (random) forces give then rise to a Brownian motion. As well known from statistical mechanics the time evolution of a many-particle Brownian motion problem is given by a Fokker-Planck type of equation. Mathematically the problem is posed by assuming a set of eigenvalues E_n depending on a variable τ . The variable τ can be seen as a 'fictitious' time, where it is important to use physical consideration to identify the parameter τ with a measurable quantity. At $\tau = 0$ the eigenvalues have the initial position

$$P(\{E_n\}, 0) = \prod_i \delta(E_i - E_{i,0}) \quad (4.2)$$

For $\tau > 0$ the system evolves according to the following Fokker-Planck equation [86] :

$$\frac{\partial P}{\partial \tau} = \sum_i^N \frac{\partial}{\partial E_i} \mu^{-1} \left(P \frac{\partial W}{\partial E_i} + \frac{\partial P}{\partial E_i} \right) \quad (4.3)$$

Here μ represents the friction coefficient. The term $W(\{E_n\})$ is given by the known total potential of the respective symmetry class.

$$W(\{E_n\}) = - \sum_{i < j}^N \ln |E_i - E_j| + \sum_i^N V(E_i) \quad (4.4)$$

We will limit our discussion to the unitary ensemble ($\beta = 2$), because the transition model is also confined to this class. This equation has the well known equilibrium distribution function at $t \rightarrow \infty$

$$P(\{E_i\}, \infty) = \prod_{i, j=0, i < j}^N (E_i - E_j)^2 \exp(-\beta \sum_{i=0}^N V(E_i)) \quad (4.5)$$

4.1.2 External Parameter Model

The crucial link between Dyson's Brownian motion model and parametric correlations consists in the relation between the 'time' τ and the external parameter. Beenakker [87, 88] observed that the initial rate of change for energy levels depending on an external parameter at small X is

$$(E_i - E_{i,0})^2 = X^2 \frac{dE_i}{dX}^2 \quad (4.6)$$

whereas the ensemble average of the Brownian motion model requires (again at small τ from [86])

$$\langle (E_i(\tau) - E_{i,0}(\tau))^2 \rangle = \frac{2\tau}{\beta\mu} \quad (4.7)$$

This implies the identification between the fictitious time τ and the perturbation parameter X :

$$\tau = X^2 \quad (4.8)$$

Dyson converted the equation above [86] into an evolution equation for the density

$$\rho(E, \tau) = \int_{-\infty}^{\infty} dE_1 \dots dE_N P(\{E_n\}, \tau) \sum_i^N \delta(E - E_i) \quad (4.9)$$

with the solution ($N = \infty$)

$$\mu \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} [\rho(E, \tau) \frac{\partial}{\partial E} (V(E) - \int_{-\infty}^{\infty} dE' \rho(E', \tau) \ln |E - E'|)] \quad (4.10)$$

Linearizing around the equilibrium ($\rho(E, \tau) = \rho_{eq}(E) + \delta\rho(E, \tau)$) gives

$$\frac{\partial}{\partial \tau} \delta\rho(E, \tau) = \frac{\partial}{\partial E} \int_{-\infty}^{\infty} dE' D(E, E') \frac{\partial}{\partial E'} \delta\rho(E', \tau) \quad (4.11)$$

$$D(E, E') = -\mu^{-1} \rho_{eq} \ln |E - E'| \quad (4.12)$$

For a constant density of states $\rho_{eq} = \rho_0$ the integral kernel is translational invariant $D(E, E') = D(E - E')$ which makes a Fourier transform suitable. In Fourier space

this equation linearizes and results in

$$\frac{\partial}{\partial \tau} \delta \rho(k, \tau) = -k^2 D(k) \delta \rho(k, \tau) \quad (4.13)$$

with the apparent solution

$$\delta \rho(k, \tau) = \delta \rho(k, 0) \exp(-k^2 D(k) \tau) \quad (4.14)$$

With the integral kernel as

$$D(k) = - \int_{-\infty}^{\infty} dE e^{ikE} \mu^{-1} \rho_{eq} \ln |E - E'| = \frac{\rho_{eq} \pi}{\mu} \frac{1}{|k|} \quad (4.15)$$

one gets

$$\delta \rho(k, \tau) = \delta \rho(k, 0) \exp\left(-\frac{\pi \rho_0}{\mu} |k| \tau\right) \quad (4.16)$$

The decoupling in Fourier space is due to the linearization procedure and allows to solve the linearized model exactly.

4.1.3 Density-Density Correlation Function

The principal interest is focussed on the parameter-dependent density-density correlation function

$$S(E', E, \tau, \tau') = \langle \sum_{i,j=1}^N \delta(E - E_i(\tau)) \delta(E' - E_j(\tau')) \rangle \quad (4.17)$$

The average $\langle \rangle$ here is the statistical average over the eigenvalues. With the natural physical assumption that the system is translationally invariant in the external quantity, i.e. $S(E, E', \tau, \tau') = S(E, E', \tau - \tau') = S(E, E', \tau)$, we can write

$$S(E, E', \tau) = \langle \rho(E, \tau) \rho(E', 0) \rangle \quad (4.18)$$

$$= \langle \delta \rho(E, \tau) \delta \rho(E', 0) \rangle + \rho_0^2 \quad (4.19)$$

The pair correlation function is at $\tau = 0$

$$S(E, E', 0) = \rho_0^2 - K(E, E') \quad (4.20)$$

Consecutively, we get

$$S(k, \tau) = S(k, 0) \exp\left(-\frac{\pi \rho_0}{\mu} |k| \tau\right) + \rho_0^2 \delta(k) \quad (4.21)$$

$$= -K(k) \exp\left(-\frac{\pi \rho_0}{\mu} |k| \tau\right) + \rho_0^2 \delta(k) \quad (4.22)$$

$$= \rho_0^2 \delta(k) + (1 - b(k)) \exp\left(-\frac{\pi \rho_0}{\mu} |k| \tau\right) \quad (4.23)$$

We will now use the external parameter representation $\tau = X^2$. Then the velocity-velocity correlation function can be expressed as

$$C(E, E', \tau, \tau') = \left\langle \sum_{i,j=1}^N \frac{\partial E_i}{\partial X} \frac{\partial E_j}{\partial X} \delta(E - E_i(X)) \delta(E' - E_j(X')) \right\rangle \quad (4.24)$$

leading to the correlator $[C(k, X) = -\frac{1}{k^2} \frac{\partial^2}{\partial X^2} S(k, X)]$

$$C(k, \tau) = \frac{1}{k^2} (1 - b(k)) [2|k| - 4|k|^2 X^2] \exp\left(-\frac{\pi \rho_0}{\mu} |k| \tau\right) \quad (4.25)$$

So far, the result is valid for the standard random matrix ensembles. In order to extend these considerations to the transition model, the idea is now to insert the function $b(k) \equiv b(k, \gamma)$ to obtain an approximation for the deformed case.

4.1.4 Decay Behavior at Large X

A simple calculation gives for $b(k) = 1 - \frac{|k|}{2\pi}$

$$S(0, X \rightarrow \infty) = \frac{1}{2\pi^2} \left(\frac{\mu}{\pi \rho_0}\right)^2 \frac{1}{X^4} \quad (4.26)$$

and

$$C(0, X \rightarrow \infty) = -\frac{1}{\pi^2 X^2} \quad (4.27)$$

Now, for $b(k) \equiv b(k, \gamma) = \frac{1}{4\pi} (|k| + 2\pi) \coth[(|k| + 2\pi)\pi/\gamma] + (|k| - 2\pi) \coth[(|k| - 2\pi)\pi/\gamma] - 2|k| \coth(|k|\pi/\gamma)$ one can approximate $b(k, \gamma)$ by the behaviour at small k (i.e. $k \ll 2\pi$), which will dominate the behaviour at large X .

$$b(k) = \frac{1}{2\pi} |k| \coth\left(\frac{|k|\pi}{\gamma}\right) \quad (4.28)$$

resulting in (for simplicity $\mu = 1$)

$$S(0, X \rightarrow \infty) = \frac{\gamma}{2\pi^3} \frac{1}{X^2} + \frac{\gamma^3}{6\pi^5} \frac{1}{X^4} \quad (4.29)$$

and

$$C(0, X) \rightarrow \infty \quad (4.30)$$

This diverging correlator represents a problem which is caused by the derivation of the two level correlation function $b(k)$. This function was calculated (see [52]) from the associated kernel, which was first used in [77]. The derivation required a $N \rightarrow \infty$ calculation of the sum over the corresponding orthogonal polynomials (compare with Ismail [91]). The usage of the $N = \infty$ kernel complicates the transition to the chaotic case. Despite the explicit dependence on the transitional parameter γ , the N dependence is vital for a smooth crossover to $\gamma = 0$. As perceivable from equation (4.28), $b(k, \gamma)$ is qualitatively different from $b(k, 0)$ easily noted by the shape of the function close to $k = 0$. The limit $N \rightarrow \infty$ transforms a smooth crossover into a sharp transition. This discrepancy is probably due to the assertion that the infinity kernel is valid in the crossover regime only. Still for smaller E , the larger k regime provides a reasonable approximation as visible from the parametric number variance. We now aim to approximate the effects for very small γ by phenomenologically adding a linear term to $b(k, \gamma)$.

$$(1 - b(k)) = \begin{cases} \frac{(1+\gamma/2)|k|}{2\pi} & k < k_t \\ \left(\frac{1}{2\pi}(|k|)\coth\left(\frac{|k|\pi}{\gamma}\right)\right) & k > k_t \end{cases} \quad (4.31)$$

Here is k_t given by

$$\frac{(1 + \gamma/2)k_t}{2\pi} = \frac{1}{2\pi} k_t \coth\left(\frac{k_t \pi}{\gamma}\right) \quad (4.32)$$

$$\approx \frac{\gamma}{2\pi^2} \quad (4.33)$$

for $\gamma \ll k_t$. Or

$$k_t = \frac{\gamma}{1 + \gamma/2} \quad (4.34)$$

Then the density-density correlation function results in

$$S(k, X) = \rho_0^2 \delta(k) + (1 - b(k)) \exp\left(-\frac{\pi \rho_0}{\mu} |k| X^2\right) \quad (4.35)$$

Consequently, integration gives then for $S(E, X) = \int S(k, X) dk$ for X large

$$\begin{aligned} S(0, X, \gamma < 1) &= \frac{1 + \gamma/2}{2\pi^2} \left(\frac{\mu}{2\pi\rho}\right)^2 \left(\frac{1}{X^4} (1 - \exp(-\mu k_t X^2)) + O(\exp(-\mu k_t X^2))\right) \\ &= (1 + \gamma/2) S(0, X \rightarrow \infty, \gamma = 0) \end{aligned} \quad (4.36)$$

i.e. for large enough X the expected decay law is recovered and provides also the first order changes for small γ . Furthermore, the scaling law proposed by Altshuler et al. [70] is recovered. The choice for the slope is justified later.

The range of validity for the hydrodynamic approximation is given by the relations $E \gg D, X \gg X_c$, where the characteristic energy scale D is given by the mean level spacing and the parameter scale X_c represents the typical oscillation frequency of the parameter evolution. This is in complete agreement with the picture of the hydrodynamic approximation as a continuum approximation, which takes the correlations between levels with large energy separation into account. This was already shown in the last chapter in equation 3.61, where the large N bulk behaviour was identical (and also universal) for different models in the same universality class. It is only natural that the inclusion of an external parameter (which is not intrinsic to the system) will not fundamentally change this view. As long as the eigenvalues are far apart, the hydrodynamic (or Coulomb gas) approximation provides reasonable results.

4.1.5 The Parametric Number Variance

The explicit dependence of the eigenvalues on the parameter makes a statistical measure desirable, which connects the regularity or chaoticity of the underlying energy spectrum with the external parameter. Such a quantity was introduced by Goldberg et al. [89]. As usual starting point serves the staircase function

$$N(E; X) = \sum_i \Theta(E - E_i(X)) \quad (4.37)$$

However, an unfolding of the spectrum is necessary to extract the general statistical behavior. Then the parametric number variance is defined

$$PNV(X) = \langle [N(E, X_0 + X) - N(E, X_0 - X)]^2 \rangle > \quad (4.38)$$

where $\langle \rangle$ is the average over intervals of length X at fixed E . The limiting behavior is $PNV(X \rightarrow 0) \rightarrow 0$. For large X , however, one has

$$PNV(E, X) \rightarrow^{X \rightarrow \infty} 2V(E) \quad (4.39)$$

where $V(E)$ is the normal number variance. The PNV can simply be calculated in the hydrodynamic approximation. We can apparently write

$$PNV(\Lambda) = 2 < [N^2(E, \Lambda) - N(E, \Lambda + \frac{\Lambda}{2})N(E, \Lambda - \frac{\Lambda}{2})] > \quad (4.40)$$

By inserting $N = \int^E \rho(E', \lambda) dE'$ and $< \rho(E_1, \lambda) \rho(E_2, \lambda) > = S(E_1, E_2, 0)$, taking into account that S is only a function of the energy and potential differences, at least in the bulk at constant density, one gets

$$< N^2(E, \Lambda) > = \int^E \int^E dE_1 dE_2 < S(E_1, E_2, 0) > \quad (4.41)$$

and further

$$< N(E, \Lambda + \frac{\Lambda}{2}) N(E, \Lambda - \frac{\Lambda}{2}) > = \int^E \int^E dE_1 dE_2 S(E_1, E_2, \Lambda) \quad (4.42)$$

From 4.21 we know S

$$S(E_1, E_2, \Lambda) = S(E_1 - E_2, \Lambda) = \rho_0^2 \int_{-\infty}^{\infty} dk e^{ik(E_1 - E_2)} (1 - b(k, q)) e^{-\frac{\pi \rho}{\gamma} |k| \Lambda} \quad (4.43)$$

giving

$$\int^E \int^{E'} dE_1 dE_2 S(E_1, E_2, \Lambda) = 4 \int dk \left(\frac{1 - \cos(k(E - E'))}{k^2} \right) (1 - b(k, q)) e^{-\frac{\pi \rho}{\gamma} |k| X^2} \quad (4.44)$$

Therefore we obtain

$$\begin{aligned} PNV(X) &= \int^E dE_1 \int^E dE_2 \int_{-\infty}^{\infty} dk e^{ik(E_1 - E_2)} (1 - b(k, q)) (1 - e^{-\frac{\pi \rho}{\gamma} |k| X^2}) \\ &= 4 \int_{-\infty}^{\infty} dk \left(\frac{1 - \cos(kE)}{k^2} \right) (1 - b(k, q)) (1 - e^{-\frac{\pi \rho}{\gamma} |k| X^2}) \end{aligned} \quad (4.45)$$

Note that for $X \rightarrow \infty$ we get $PNV(X \rightarrow \infty) = 2\Sigma^2(E)$ which is the known limit. From 4.45 it is obvious that the limiting behavior for $X \rightarrow 0$ is $PNV(X) \propto X^2$. This clearly differs from the predicted behavior by Faas et al. [92, 71], where a linear slope was calculated, revealing a weakness of the hydrodynamic approximation, which is problematic at small X and small E when the levels are too close represent a Coulomb fluid anymore. The predicted behavior is reasonable and agrees also with numerical calculations [71]. Note that the integration at small k is no longer relevant here, as the PNV is a statistic over the entire spectrum and quantifies integral behavior. As apparent from figure 4.1, the actual behaviour deviates from the Wigner-Dyson case by exhibiting an increased value for the parametric number variance. Note that this deviation can be nicely explained by the results from the present transition model.

4.2 Exact Treatment of the Parametric Correlation Model

4.2.1 Ansatz for the Density-Density Correlation Function

As discussed before, since the hydrodynamic solution has its limitations, a more exact treatment of the problem is useful. Starting out from the fundamental equation 4.3 the exact solution for the undeformed case ($\gamma = 0$) was calculated by Beenakker and Rejaei [88]. The derivation uses a mapping from the N -level Fokker-Planck equation to a decoupled N -level Schrödinger equation ($\beta = 2$).

$$\mu \frac{\partial \Psi(\{x_i\})}{\partial \tau} = \left\{ \frac{1}{2} \sum_{i=0}^{N-1} \frac{\partial^2}{\partial E_i^2} - 2 \sum_{i=0}^{N-1} E_i^{N-1} + N \right\} \Psi(\{x_i\}) \quad (4.46)$$

The single level eigenfunctions of the static N -level system can be calculated exactly, and a natural solution can be found by expanding the multibody dynamic 'wavefunction' in terms of the single level eigenfunctions. As a well established procedure this yields

$$S(E, E', X) = \sum_{n=0}^{N-1} \sum_{m=N}^{\infty} H_n(E) H_n(E') H_m(E) H_m(E') \exp(\omega(n-m)X^2) \quad (4.47)$$

where ω is the oscillator frequency. The generalization can be justified by noting that the many-body wave function can be written as a Vandermonde determinante which

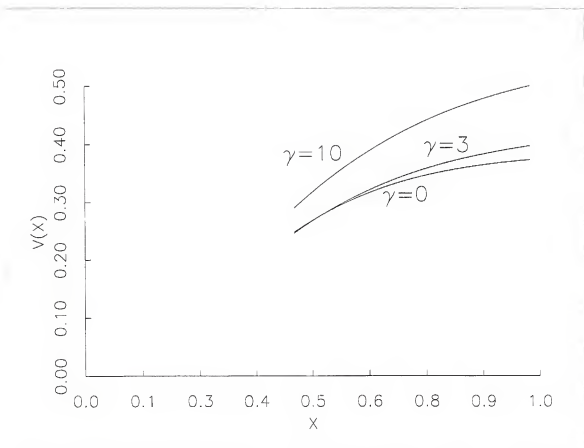


Figure 4.1: Parametric number variance

is a product of single particle wave functions. This implies that the density-density correlation functions can be formulated in analogy to equation 4.47. Then we arrive at the following density-density function for $\gamma > 0$.

$$S(E, E', X) = \sum_{n=0}^{N-1} \sum_{m=N}^{\infty} h_n(E, q) h_n(E', q) h_m(E, q) h_m(E', q) \exp((n_q - m_q) \omega X^2) \quad (4.48)$$

It follows from 4.47 by replacing the polynomials $h_n(x)$ and the eigenvalues $n + \frac{1}{2}$ by their natural q extensions $h_n(x, q)$ and $n_q + \frac{1}{2}$ [91]. This expression shows the desired behavior:

In the limit $X = 0$ the known result from the theory of orthogonal polynomials is recovered. i.e.

$$\begin{aligned} S(E, E') &= R_2(E, E') = \sum_{n=0}^{N-1} \sum_{m=N}^{\infty} h_n(E, q) h_n(E', q) h_m(E, q) h_m(E', q) w(E) w(E') \\ &= \sum_{n=0}^N h_n^2(E, q) w(E) \delta(E - E') - \left(\sum_{n=0}^N h_n(E, q) h_n(E', q) \sqrt{w(E) w(E')} \right)^2 \\ &= \sigma(E) \delta(E - E') - K^2(E, E') \end{aligned} \quad (4.49)$$

Here $R_2(E, E')$ represents the two level (static) correlation function, $\sigma(E)$ the level density and $K(E, E')$ the kernel. The eigenvalue $n_q = \frac{1-q^n}{1-q}$ appears in the formal single level q -Greensfunction

$$G_s(E, E', X) = \sum_{n=0}^{\infty} h_n(E, q) h_n(E', q) \exp(-\omega n_q X^2) \quad (4.50)$$

which is the solution of the one level eigenvalue equation. Therefore these two limits for the density-density correlation function are exact.

This illustrates that equation 4.48 denotes a possible q -generalization of the density-density correlation function in 4.47.

This q -generalization will not be unique, though, as it is well known that many possible q -Hermite polynomials exist. A more fundamental derivation of equation 4.48 based on a generalized Fokker-Planck equation has eluded us so far.

4.2.2 Large X Behavior of S

In order to understand the dynamic properties of the transition model, the expression 4.48 has to be analyzed for large X. Starting from the density-density correlation function

$$S(x, x', \tau) = \sum_{m=0}^{N-1} \sum_{n=N}^{\infty} h_n(x, q) h_n(x', q) h_m(x, q) h_m(x', q) w_1(x, q) w_1(x', q) e^{(m_q - n_q) \omega X^2} \quad (4.51)$$

we can choose $E = E' = 0$. Then one can calculate the values for the q-Hermite polynomials by simply applying $h_{2n}(x, q) = 2x h_{2n-1}(x, q) - \frac{1-q^{2n-1}}{q^{2n-1}} h_{2n-2}(x, q)$.

$$h_{2n}(0) = \frac{[\prod_i^n (1 - q^{2i-1})]^2}{q^{n^2}} \quad (4.52)$$

The Hermite polynomials with odd numbers vanish at zero argument. Including the normalization one can write then

$$h_{2n}(0) h_{2n}(0) = \frac{[\prod_i^n (1 - q^{2i-1})]^2}{q^{-n} \gamma(q, q)_{\infty} \prod_{i=0}^{2n} (1 - q^{i+1})} \quad (4.53)$$

The potential can be written as

$$w(E) = \frac{1}{(-q \exp(2x); q)_{\infty} (-q \exp(-2x); q)_{\infty}} \quad (4.54)$$

$$= 2q^{-1/4}(q; q)_{\infty} \sqrt{\frac{\gamma}{\pi}} e^{-E^2/\gamma} \frac{\cosh(E)}{\Theta_4(\pi E/\gamma; \exp(-\pi^2/\gamma))} \quad (4.55)$$

which gives for $E=0$

$$w(0) = \frac{2}{\sqrt{\pi}} q^{-1/4}(q; q)_{\infty} \sqrt{\gamma} \frac{1}{1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp(-n^2 \pi^2/\gamma)} \quad (4.56)$$

It follows

$$S(X, q) = A c \sum_{m=N}^{\infty} \sum_{n=0}^{N-1} \frac{[\prod_i^n (1 - q^{2i-1})]^2}{q^{-n} \gamma(q, q)_{\infty} \prod_{i=0}^{2n} (1 - q^{i+1})} \frac{[\prod_i^m (1 - q^{2i-1})]^2}{q^{-m} \gamma(q, q)_{\infty} \prod_{i=0}^{2m} (1 - q^{i+1})} \exp(\omega(m_q - n_q) X^2) \quad (4.57)$$

Here the constant c will be chosen so that the density is given by a constant independent of γ and N. The frequency ω is given by $\omega = c/\mu$ (see appendix B) and the

potential term A is

$$A = \frac{2q^{-1/4}}{\pi} \frac{1}{1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp(-n^2 \pi^2 / \gamma) [(q; q)_{\infty}]^2} \quad (4.58)$$

The constant A ($A \approx 2/\pi$), the first sum and the second sum can be evaluated for small γ , i.e. $\gamma N \ll 1$. The term inside the sums can be estimated for small γ

$$g_n \equiv \frac{[\prod_{i=0}^n (1 - q^{2i-1})]^2}{\prod_{i=0}^{2n} (1 - q^{i+1})} \quad (4.59)$$

$$= \frac{(1-q)(1-q^3) \cdot \cdot (1-q^{2n-1})}{(1-q^2)(1-q^4) \cdot \cdot (1-q^{2n})} \quad (4.60)$$

with

$$\frac{1 - q^{2n-1}}{1 - q^{2n}} = 1 - \frac{e^{-\gamma(2n-1)}(1 - e^{-\gamma})}{1 - e^{-2n\gamma}} \quad (4.61)$$

$$\approx^{\gamma \ll 1} 1 - \gamma(1 + \gamma/2) \frac{e^{-2\gamma n}}{1 - e^{-2\gamma n}} \quad (4.62)$$

$$\approx^{n\gamma \ll 1} (1 + \gamma) \left(1 - \frac{1 - \gamma/2}{2n}\right) \quad (4.63)$$

and

$$g_n = (1 + \gamma)^n \prod_{k=1}^n \left(1 - \frac{1 - \gamma/2}{2k}\right) \quad (4.64)$$

The logarithm of g can be evaluated

$$\ln(g_n) = n \ln(1 + \gamma) + \sum_{k=1}^n \ln\left(1 - \frac{1 - \gamma/2}{2k}\right) \quad (4.65)$$

$$\approx -\frac{\gamma^2 n}{2} - \ln(n^{\frac{1}{2}(1 - \frac{\gamma}{2})}) \quad (4.66)$$

resulting in

$$g_n \approx^{n\gamma \ll 1} \frac{\exp(-\frac{\gamma^2 n}{2})}{n^{\frac{1}{2}(1 - \gamma/2)}} \quad (4.67)$$

By neglecting terms of smaller order in $n\gamma$, one arrives finally at

$$S(X) = \frac{2c}{\pi\gamma} \sum_{m=0}^{N-1} q^m g_m e^{\omega m q X^2} \sum_{n=N}^{\infty} q^n g_n e^{-\omega n q X^2} \quad (4.68)$$

We are able to determine the constant c by demanding a constant density σ . Since we can calculate the density only at $E = 0$ exactly, we will choose c so that the density is unity at that point. This assumption is clearly justified by looking at the figures 3.1 and 3.2, which illustrate that the density does not significantly change close to the origin.

A short calculation gives (after approximating g_k)

$$\sigma_N(0) = \frac{2\sqrt{c\pi}}{\sqrt{\gamma}} \sum_{k=0}^{N-1} \frac{e^{-\gamma k}}{\sqrt{2k_q}} \quad (4.69)$$

$$\approx \frac{\sqrt{2c\pi}}{\sqrt{\gamma}} \sqrt{(N)_q} \quad (4.70)$$

The condition $\sigma_N(0) = \text{const.} = 1$ leads to

$$c = \frac{\gamma\pi^2}{2N_q} \quad (4.71)$$

$$\omega = \frac{\pi^2}{2\mu N_q} \quad (4.72)$$

By replacing the sums by integrals

$$\sum_{m=0}^{N-1} q^m g_m e^{\omega m_q X^2} \sim \int_0^N \frac{q^m}{\sqrt{m_q}} e^{m_q \omega X^2} \quad (4.73)$$

and substituting $m_q \rightarrow m$ we arrive at

$$S(X) = \frac{2}{N_q} \int_0^{N_q} \frac{e^{m\omega X^2}}{\sqrt{m}} dm \int_{N_q}^{\infty} \frac{e^{-n\omega X^2}}{\sqrt{n}} dn \quad (4.74)$$

After a further substitution step ($s^2 = m$) and insertion of the relations for c and ω one obtains

$$S(X) = \frac{1}{2} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt e^{(s^2 - t^2) X^2 / \mu} \quad (4.75)$$

This equation represents the generalization to the standard expressions in [88] and [70] for disordered systems.

The equation 4.75 can be used to determine the large X behaviour. Starting by calculating the two integrals as

$$\begin{aligned}\int_0^{N_q} \frac{e^{s\omega X^2}}{\sqrt{s}} ds &= \frac{1}{\sqrt{\omega X^2}} \int_0^{N_q \omega X^2} \frac{e^{s'}}{\sqrt{s'}} ds' = \frac{1}{\sqrt{N_q \omega X^2}} e^{N_q \omega X^2} \\ \int_{N_q}^{1/\gamma} \frac{e^{-t\omega X^2}}{\sqrt{t}} dt &= \frac{1}{\sqrt{\omega X^2}} \int_{N_q \omega X^2}^{\omega X^2/\gamma} \frac{e^{-t'}}{\sqrt{t'}} dt' = -\frac{\sqrt{\gamma}}{\omega X^2} e^{-\omega X^2/\gamma} + \frac{1}{\sqrt{N_q \omega X^2}} e^{-N_q \omega X^2}\end{aligned}$$

where the large argument expansion for the incomplete Gamma function was used ($\gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt \sim x^{a-1} e^{-x}$). Multiplying the two integrals as well as inserting c and ω results in

$$S(X) = \frac{\mu^2}{2\pi^4 X^4} (1 - \sqrt{1 - \exp(-\gamma N)}) e^{-X^2 \exp(-\gamma N) \pi^2 / 2\mu} \quad (4.76)$$

This expression has for $X^2 \gg \mu e^{\gamma N}$ the known limit $S(X) = \frac{\mu^2}{X^4}$, which was already obtained from the hydrodynamic approximation 4.29. However, for $X^2 \ll \mu e^{\gamma N}$ the correlation function decays with the markedly different behavior

$$S(X) = \frac{\mu}{\pi^2 X^2} e^{-\gamma N} \quad (4.77)$$

This result demonstrates the breakdown of the universal decay law for sufficiently large, but not too large, X ($\mu \ll X^2 \ll e^{-\gamma N}$). Equation 4.76 contains not only the expected general decay term $\propto \frac{1}{X^4}$, but also an exponential correction. This additional term is quite small at large X and small γN , but for larger γN it will lead to weaker decay behavior at large X compared to the universal result.

For small X , which is defined by the relation $X^2 \ll \mu$, the density-density correlation reads

$$S(X) = \frac{\sqrt{2\mu}}{4\sqrt{\pi} X} \left[\operatorname{erf}\left(\sqrt{\frac{X^2 \pi^2}{2\mu \gamma N_q}}\right) - \operatorname{erf}\left(\sqrt{\frac{X^2 \pi^2}{2\mu}}\right) \right] \quad (4.78)$$

in terms of the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$. By distinguishing the two cases $X^2 \pi^2 \ll 2\mu \gamma N_q$ and $X^2 \pi^2 > 2\mu \gamma N_q$, $X^2 \pi^2 \ll 2\mu$ one can easily understand the small X behavior. In the case $X^2 \pi^2 \ll 2\mu \gamma N_q$ the expansion of the error functions

leads to

$$S(X) = \frac{\sqrt{\pi}}{4} \left[\frac{1}{\sqrt{\gamma N_q}} - 1 \right] + O(X) \quad (4.79)$$

On the other hand we have for $X^2 \pi^2 > 2\mu\gamma N_q$, $X^2 \pi^2 \ll 2\mu$

$$S(X) = \frac{\sqrt{2\mu}}{4\sqrt{\pi}X} \left[1 - \frac{\sqrt{2\mu\gamma N_q}}{\pi^{3/2}X} e^{-\frac{X^2 \pi^2}{2\mu\gamma N_q}} \right] + O(X^0) \quad (4.80)$$

This result points out the special nature of the value $X=0$. For small, but not too small values of X the density correlator has a large term of the order of $\frac{1}{X}$. At very small X , however, the density correlator converges to the value $\frac{1}{\sqrt{\gamma N_q}} - 1$. This clearly describes a peak at the origin. In case of the undeformed model we have $\gamma N_q = 1 - e^{-\gamma N} = 0$ and therefore a divergent value at $S(X \rightarrow 0)$. A variation of the above calculation obtains indeed

$$S(X \rightarrow 0, \gamma = 0) = \frac{\sqrt{2\mu}}{4\sqrt{\pi}X} \quad (4.81)$$

indicating the diverging result. The calculation demonstrates that the divergence in the case of vanishing γ broadens into a wider and finite peak for increasing disorder. The interpretation of this phenomenon is connected to the presence of level self-correlations. The level repulsion excludes levels at the same energy, but the statistical fluctuation causes interference of the same level at the same energy. This contribution appears naturally in the parameter-independent density-density correlation function $S(E, E')$ as a δ -function (equation 4.21 and also 4.57). The finite distance between the levels in the transition model prohibits on the other hand the strong self-correlations, so that the correlation function converges to a large, but finite value (Here we assume $\gamma N \ll 1$).

The phenomenon of the broadening of a correlation function at small arguments appears also in the velocity-velocity correlation function which we will discuss next.

4.2.3 The Velocity Correlator

In the Fokker-Planck formulation of equation 4.3 the density-density correlation function is the fundamental quantity. However, from physical consideration the velocity correlator is of equal importance.

The velocity correlator can directly be derived from the density-density correlation function as

$$C(E, E', X, X') = \frac{\partial^2}{\partial X \partial X'} \int dE \int dE' S(E, E', X, X') \quad (4.82)$$

The evaluation of the complete function $C(E, E', X, X')$ is technically difficult, since a large N approximation for the q -Hermite polynomials, which partially constitute the eigenfunctions, for all arguments is presently not available. However, in analogy to the density-density correlation function, it is possible to calculate the velocity correlator at vanishing X . The technical procedure is outlined in appendix B with the result

$$C(X, X') = \frac{Ac}{2\gamma} \left[\sum_{m=0}^{[N/2]} \sum_{n=[N/2]+1}^{\infty} \frac{q^{2n+2m} ((2n+1)_q!)^2 ((2m-1)_q!)^2}{(2n+1)_q! (2m)_q! ((2n-1)_q - (2m)_q)^2} \right] \quad (4.83)$$

$$\frac{\partial^2}{\partial X \partial X'} e^{((2m)_q - (2n+1)_q)\omega(X-X')^2} \quad (4.84)$$

$$+ \sum_{m=1}^{[N/2]-1} \sum_{n=[N/2]+1}^{\infty} \frac{q^{2n+2m} ((2n-1)_q!)^2 ((2m+1)_q!)^2}{(2n)_q! (2m+1)_q! ((2n-3)_q - (2m)_q)^2} \quad (4.85)$$

$$\frac{\partial^2}{\partial X \partial X'} e^{((2m)_q - (2n-3)_q)\omega(X-X')^2} \quad (4.86)$$

for $N > 2$. Since the sum representation is difficult to evaluate, the approximation by an integral is desired. As outlined in appendix B, we get

$$C(X, X') = \frac{1}{\mu} \left[\int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N q}}} dt \frac{s^2 + t^2}{s^2 - t^2} e^{(s^2 - t^2)(X - X')^2 / 2\mu} \right] \quad (4.87)$$

$$- \frac{\pi^2 (X - X')^2}{2\mu} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N q}}} dt (t^2 + s^2) e^{(s^2 - t^2)(X - X')^2 \pi^2 / 2\mu} \quad (4.88)$$

From here on we use $X \equiv X - X'$. This formula can be evaluated for small and large X . For large X we write the first integral as

$$C(X) = \frac{1}{\mu} \int^{\pi^2 X^2 / 2\mu} d\left(\frac{\pi^2 X^2}{2\mu}\right) \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N q}}} dt (s^2 + t^2) e^{-(s^2 - t^2) X^2 \pi^2 / 2\mu} \quad (4.89)$$

$$(4.90)$$

In both integrals appears now the same expression. Concentrating only on terms which are relevant for $\frac{\pi^2 X^2}{2\mu} \gg 1$ we obtain as an intermediate result

$$\int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt (t^2 + s^2) e^{(s^2 - t^2) X^2 \pi^2 / 2\mu} \quad (4.91)$$

$$= \frac{\mu^2}{\pi^4 X^4} \left[1 - \left(\sqrt{\gamma N_q} + \frac{1}{\sqrt{\gamma N_q}} \right) e^{-\frac{\pi^2 X^2}{2\mu} (\frac{1}{\gamma N_q} - 1)} + O(1/X^4) \right] \quad (4.92)$$

Inserting this result in equation 4.87 and evaluating the parameter integration in $\frac{\pi^2 X^2}{2\mu}$ leads then to

$$C(X) = -\frac{1}{\pi^2 X^2} + \frac{(\sqrt{\gamma N_q} + \frac{1}{\sqrt{\gamma N_q}})}{2\pi^2 X^2} e^{-\frac{\pi^2 X^2}{2\mu} (\frac{1}{\gamma N_q} - 1)} + O(1/X^4) \quad (4.93)$$

This result recovers the universal velocity correlator in the accordingly called 'universal' regime, i.e. $C(X) = -\frac{1}{\pi^2 X^2}$. Our result also conveys that even at $\gamma N \neq 0$ we reestablish the universal correlator for sufficiently large X , in particular at $\frac{\pi^2 X^2}{2\mu \gamma N_q} \gg 1$. It is interesting to note here that this universal decay behavior can be also recovered from the completely chaotic Sinai billiard by use of the Gutzwiller trace formula [92]. Furthermore, it is obvious that the correction term in γN leads to a smaller velocity correlator. This is in perfect agreement with the result for the density-density correlation function 4.75 as well as with the intuitive picture that the deformation in the transition model weakens the correlations. Even though this is directly apparent only from the energy level point of view, the parametric correlations will show the same tendency as they are tightly coupled to the energy eigenlevel behaviour. In particular, for the Poisson case one would expect a velocity correlator identically zero. However, the weak exponential correction term is unlikely to be discovered in experimental or numerical work.

For small X the velocity correlator has a nonuniversal behavior. As the transitional model possesses a new scale γN , the correlator will exhibit new features at small X . In writing the expression 4.75 as (take $\eta = \frac{\pi^2}{2\mu}$)

$$C(X) = \frac{1}{\mu} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt [e^{(s^2 - t^2) X^2 \eta} + \frac{2s^2}{t^2 - s^2} - 2s^2 \int_0^\infty d\lambda e^{\lambda \eta (s^2 - t^2) X^2}] \quad (4.94)$$

$$-2\eta X^2 s^2 e^{\eta(s^2-t^2)X^2} + 2\eta X^2 t^2 e^{\eta(s^2-t^2)X^2}] \quad (4.95)$$

the five individual terms can be analyzed separately. Neglecting for $\eta X^2 \ll 1$ the s integral, the first and last terms are

$$\begin{aligned} T1 &= \frac{1}{2\sqrt{\eta X^2}} [\gamma(1/2, \frac{\eta X^2}{\gamma N_q}) - \gamma(1/2, \eta X^2)] \\ T5 &= \frac{1}{\sqrt{\eta X^2}} [\frac{1}{2}\gamma(1/2, \frac{\eta X^2}{\gamma N_q}) - \frac{1}{2}\gamma(1/2, \eta X^2) + \sqrt{\eta X^2} \gamma N_q e^{-\frac{\eta X^2}{\gamma N_q}} - \sqrt{\eta X^2} e^{-\eta X^2}] \end{aligned}$$

In the approximation chosen the first and the third term are similar and combine with the fifth term to

$$(\frac{1}{\sqrt{\gamma N_q}} e^{-\frac{\eta X^2}{\gamma N_q}} - e^{-\eta X^2}) + O(\sqrt{X^2 \eta}) \quad (4.96)$$

The fourth term can be shown as small in $O(\sqrt{X^2 \eta})$, but the third term obviously contributes a constant term.

$$T3 = 1 + \frac{1}{\gamma N_q} [\frac{1}{2}(1 - \gamma N_q) \ln(\frac{1 + \sqrt{\gamma N_q}}{1 - \sqrt{\gamma N_q}}) - \sqrt{\gamma N_q}] \quad (4.97)$$

The result can be written as

$$C(X) = \frac{1}{\mu} [(\frac{e^{-\frac{\eta X^2}{\gamma N_q}}}{\sqrt{\gamma N_q}} - e^{-\eta X^2}) + \frac{1}{\gamma N_q} [\frac{1}{2}(1 - \gamma N_q) \ln(\frac{1 + \sqrt{\gamma N_q}}{1 - \sqrt{\gamma N_q}})]] \quad (4.98)$$

The entire correlator is depicted in figure 4.2, calculated from the straightforward integration of the integrals 4.75.

It is apparent from this expression that the velocity correlator exhibits a peak around $X=0$. The width of the peak is of the order γN , which approaches zero in the weakly disordered case. In this limit the peak diverges and becomes a δ -function. This can be seen in the limit of $\gamma N \rightarrow 0$, where the only term remaining is the nonanalytic term $C(X) = \frac{1}{\sqrt{\gamma N_q}} e^{-\frac{\eta X^2}{\gamma N_q}} + O(const.)$. This term describes the divergence as well as its width. The peak is visible in figure 4.3 for a small γN .

At $\gamma N = 0$ the limit $X \rightarrow 0$ is not well defined, as it is clear from equation 4.75 that the limiting procedure $C(X \rightarrow 0, \gamma N = 0)$ will lead to $C(X = 0, \gamma N = 0) = 0$.

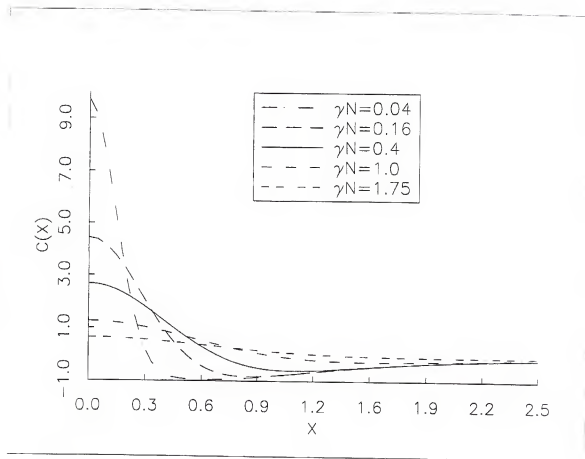


Figure 4.2: Unscaled Velocity Correlator for five different values of γ

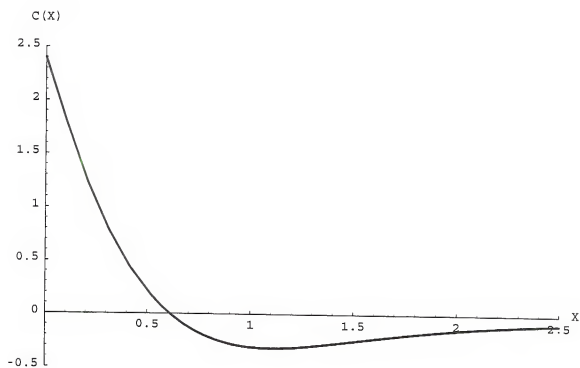


Figure 4.3: Peak structure in level sensitivity

This is an explicit violation (quite apparent from figure 4.3 without peak) of the sum rule

$$\int_0^\infty C(X) dX = 0 \quad (4.99)$$

which is an obvious conclusion from equation 4.82. The δ -function peak at $X=0$ has to be accounted for the Wigner-Dyson limit.

The problem to evaluate the velocity correlator for $q=1$ at $X=0$ was already recognized by Simons and Altshuler [70]. They suggested a level-broadening procedure, where the statistical averaging is not performed over sharp (δ -function) energy levels, but over slightly smeared levels (a gaussian broadening function was used in [70]). This broadening leads to a well defined as well as positive $C(0)$ and subsequently to compliance with the sum rule. This procedure is physically justified, since the energy levels show nonzero equilibrium fluctuations.

Note that the effect of the broadening procedure on the velocity correlator is quite similar to the inclusion of disorder in the transition model. A singularity at $X=0$ is smoothened out into a finite peak as a function of a parameter which determines the width of the peak. It would certainly be quite interesting to consider the level broadening procedure also for the transitional model in order to include the universal (weakly disordered) regime. However, this would pose strong technical problems, as the 'smoothened' ensemble average over a q -Hermite ensemble is not clear. Therefore, we will restrict here the physical validity to $\gamma N \geq 1$. In this regime the peak is determined by the disorder parameter γN , because the characteristic energy scale, the mean level spacing, is set to one.

4.2.4 Scaling of the Velocity Correlator

The universal behavior of random matrix results at $\gamma = 0$ is assumed to appear also in results with parametric correlations. However, the additional variable (the external parameter X) will result in the occurrence of a new characteristic scale in X ,

which can be thought to be analogous to the mean level spacing. For the parametric correlations the velocity correlator at zero field is a suitable description (see [68]). Moreover, after rescaling with the characteristic quantity, the correlator becomes independent of the physical properties of the involved system and displays universal behavior (see [68, 69]). This was even claimed for weakly disordered or not completely chaotic systems, so that a comparison to the presented theory is appropriate. We will limit our discussion here to disordered systems as the theory is more developed on this. For this investigation one can first observe that the rescaling

$$X \rightarrow x = \sqrt{C(0)}X \quad (4.100)$$

$$c(x) = C(x)/C(0) \quad (4.101)$$

is performed by a constant appropriately chosen by the zero argument correlator. The significance of the zero velocity correlator resides also in the fact that it can be identified with the conductance in the case $\gamma = 0$. The assumption consists in the applicability of a fluctuation-dissipation theorem for the random matrix ensembles. One indicator is already the presence of a Fokker-Planck equation which is a fundamentally nonconservative equation. The occurrence of a dissipative system indicates already that usual transport relations can be applied. The validity of the theorem requires that the external perturbation remains small, which will only affect the particles in the medium considered but not the medium itself. The system reacts to the external perturbation in sole dependence on the equilibrium fluctuations. This necessitates to formulate the dissipation-fluctuation theorem in the statistical context used so far. The averaged energy is $\epsilon(X) = \langle \rho(X) \rangle$ where the brackets indicate a statistical average over the distribution $P(\{E_n\}, X)$, and ρ is given by equation 4.9. One can then assume $\frac{\partial \epsilon(X)}{\partial X} = 0$ without loss of generality. The fluctuation-dissipation theorem can be written as [69]

$$\frac{\partial \epsilon}{\partial t} = \alpha \left(\frac{\partial \epsilon}{\partial X} \right)^2 \left(\frac{\partial X}{\partial t} \right)^2 \quad (4.102)$$

where $(\frac{\partial \epsilon}{\partial X})^2 = \lim_{X' \rightarrow X} < \sum_{ij} \frac{\partial E_i}{\partial X} \frac{\partial E_j}{\partial X} \delta(E - E_i(X)) \delta(E - E_j(X')) >_{E=E'=0}$, and α is a constant. A similar relation was originally proposed by Wilkinson [93, 94]. The derivative in the external parameter of the energy corresponds to the velocity correlator $C(X)$, and the equation above can be written as [69]

$$\frac{\partial \epsilon}{\partial t} = \pi \hbar C(0) \left(\frac{\partial X}{\partial t} \right)^2 \quad (4.103)$$

This allows to identify the velocity correlator at $X = 0$ with the conductance describing the sensitivity of the spectrum to changes in the external parameter X . This identification was in another context also derived in reference [95] where the different nature between the dissipative conductance given by the Kubo formula and the conductance due to curvature of the level at the origin $C(0)$ was pointed out. The equality between those two quantities is only given for system sufficiently close to the the universal (metallic) regime.

Even though the Thouless argument is questionable at larger disorder, we can still demonstrate the breakdown of the universality by scaling the velocity correlator by the value $C(0)$. The result is depicted in figure 4.4.

From the expression 4.98 and 4.93 the scaling procedure yields for the transition model at small arguments, i.e. $\frac{x^2}{\mu \gamma N_q} \ll C(0)$

$$c(x) = 1 - \frac{\sqrt{\gamma N_q} (1 + \gamma N_q - 2(\gamma N_q)^{3/2})}{(1 - \gamma N_q) \ln[(1 + \sqrt{\gamma N_q}) / (1 - \sqrt{\gamma N_q})]} x^2 \quad (4.104)$$

whereas for large argument $(\frac{x^2}{\mu \gamma N_q} \gg C(0))$

$$c(x) = -\frac{1}{\pi^2 x^2} \left[1 - \frac{\sqrt{\gamma N_q} + \frac{1}{\sqrt{\gamma N_q}}}{2} e^{-\frac{x^2}{i \ln[(1 + \sqrt{\gamma N_q}) / (1 - \sqrt{\gamma N_q})]}} \right] \quad (4.105)$$

Note that for all γN the universality still holds true for large enough x , where the correction term becomes exponentially small. For smaller x , however, the strong disorder parameter can not be scaled away, and the universality in this regime breaks down. This breakdown is associated with the breakdown of the universality of spectral

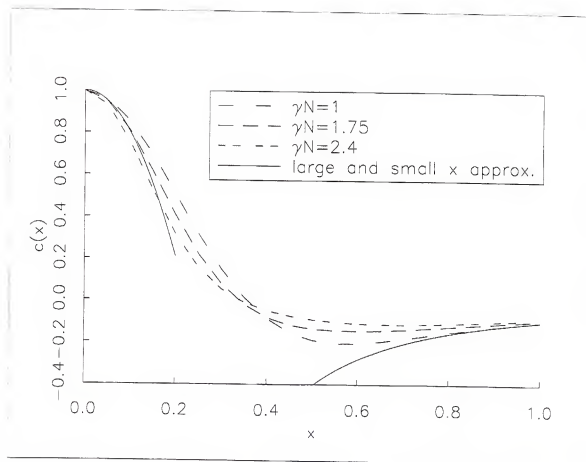


Figure 4.4: Scaled Velocity Correlator for three different values of γ

correlations at strong disorder [52]. The numerical results from figure 4.4 support that conclusion. The scaled velocity correlator for the largest value of $\gamma = 2.4$ is clearly not fitting the known asymptotics anymore. In particular, for small X , the $\gamma = 2.4$ curve lies below the asymptotic behavior even though the third correction in the asymptotic curve is positive.

These predictions should be possible to verify, for example by comparing the behavior with numerical results for the tight binding Anderson model for strong disorder near the critical regime.

CHAPTER 5 DISCUSSION

This chapter recapitulates the basic transitional statistical model at the core of this work and restates the central results of the approach. These results are then compared with actual expectations from physical systems, and several predictions are drawn. The beneficial contributions of the model will become clear in a detailed comparison with other approaches to the transitional statistics problem. Finally an outlook will be presented focussing on the future of this physical area and on additional possible model developments.

5.1 The Transitional Statistics Model

Complex quantum systems exhibit certain common characteristics. The analysis of their energy levels often reveals a similar universal behavior. Standard random matrix theory has been instrumental in explaining general level statistic behavior in chaotic and metallic systems. It is based on the underlying assumption that the energy levels of a complex system, about which we have limited knowledge, are better described by a Hamiltonian matrix, which is as random as possible and subject to certain global symmetry conditions like hermiticity.

The great success of random matrix theory for otherwise difficult to approach systems and the possibility to connect phenomena from seemingly different areas in physics has made it a widely accepted and helpful interdisciplinary theoretical tool in physics.

Many of the areas implied contain also physical systems, which are neither absolutely chaotic nor completely ordered and, therefore, show a different level behavior. The

characteristics of these systems are in between the random matrix theory and a theory of complete level randomness (Poisson statistics). Of most interest are systems, whose characteristics change as function of a certain intrinsic parameter, which determines its degree of chaos or disorder. By varying this parameter the system crosses over from a chaotic spectrum to a regular (integrable) spectrum.

To understand these transitional systems better, it is a well motivated idea to develop a new extended random matrix theory, which can account for the change in behavior. The model described in this work represents a contribution in this direction. Its goal is not only to explain already observed spectra, but also to predict general features for systems in spectral transition.

The model can intuitively most easily be understood in terms of the two-dimensional Coulomb gas model, which replaces (in an exact mapping) the rather abstract energy levels by two-dimensional particles (electrons). The particles repel each other, but they do not escape from each other due to the presence of a confining potential. The repelling force has been identified as the source of the most fundamental effect, the level repulsion. As the system changes, the level repulsion is still present, albeit significantly less pronounced.

Therefore it is natural to keep the two particle repelling potential and to generalize the confining potential. Standard random matrix theory features a very strongly confining potential, which keeps the eigenvalues close together. This potential has the characteristic behavior to confine the particles to a finite distance from the origin, even when the number of particles diverges in the thermodynamic limit. This property leads obviously to a vanishing distance between particles for $N \rightarrow \infty$, which explains the picture of the particle system as compressible fluid.

The system does not exhibit compressibility in case of a weakly or softly confining potential, as the particles in the thermodynamic limit move towards infinity instead of being compressed. The system is therefore incompressible, and one can expect a

qualitatively different behavior.

The justification for changing the confining potential can be found within the maximum entropy approach framework. It states that the potential is determined by using the physical eigenvalue density as input. As the density in both random matrix ensembles are constant, the transitional model complies with the general approach. The model can be exactly solved in terms of orthogonal polynomials, which certainly represents a strength of this formulation. It also introduces one parameter governing the transition from a chaotic to a regular spectrum.

The results of any transitional model have to be carefully evaluated. The problem lies in the fact that directly observable and predicted quantities such as the unscaled correlation functions are difficult to compare. Therefore, the usual way of comparison is to establish a statistical analysis, in which the quantities are rescaled so that the level density becomes constant. Consequently, the fluctuations gain universal meaning and can be interpreted. These statistical results can then be compared with either numerical or experimental data. Any transitional model should be able to recover the two fundamental statistical limits, the Poisson statistics on the one hand and the Wigner-Dyson statistics on the other hand. Furthermore a model should be capable of describing the crossover between these two limits as a function of the transition parameter.

The transitional model evaluated herein possesses all those features. In particular, the two-level correlation function $R_2 = 1 - K^2$ has an explicit crossover into the well known Wigner-Dyson correlation function [4] for the parameter limit $\gamma \rightarrow 0$. The other limit is obtained by imposing $\gamma \rightarrow \infty$. The two-level correlation function becomes then $R_2 = 1$, as expected from Poisson statistics. However, this limit is discontinuous, as it can be seen from the fact that for all $\gamma < \infty$ we have $R_2(0) = 0$, while for $\gamma = \infty$ the value is $R_2(0) = 1$. This is a remnant of the fact, that the transitional model contains still the two body repulsive force, i.e. two levels are

disfavored to be at the same energy. In addition, it is quite interesting to note the fundamentally different decay behavior of R_2 , indicated by the change from a power law to exponential.

The nearest neighbour spacing distribution is a measure which shows a distinct difference between Wigner-Dyson and Poisson statistics. The result of the model in figure 3.5 interpolates smoothly between these two extreme cases. The maximum present in the Wigner-Dyson spacing moves for larger transition parameter towards the origin and joins in the limit $\gamma \rightarrow \infty$ into the exponential decay of a Poisson spacing. Two important observations are: First, the slope of the nearest neighbour spacing distribution at the origin can be shown to directly depend on the symmetry class for standard random matrices 1.32. The obvious deviation from this general behavior emphasizes the presence of a new (universality) class of random matrices in the transitional model. Secondly, all the different curves intercept at one point in figure 3.5. This striking feature could imply a scaling point and therefore a new statistics accompanying the model.

Of vital interest are also long-range integrated statistical properties. The usual measures of this behavior are the number variance and the Δ_3 -statistics. The difference between the two limiting statistics is substantial, as the Wigner-Dyson model implies a slow logarithmic increase in both measures, indicating a stiffness of the spectrum due to the level repulsion compared to the linear increase in the Poisson statistics. The transitional model recovers both extremal statistics as well as interpolates between these two curves for intermediate values of the transitional parameter.

The result for the Δ_3 -statistics displayed in figure 3.6 supports that claim. The analysis of these measures indicates that the statistics can be written in the form

$$\Delta_3(N) \approx a \ln(2\pi N) + bN \quad (5.1)$$

The linear term in N appears only in the transitional model and will dominate the Δ_3 -statistics or number variance for sufficiently large N . The slope of the linear increase,

the constant b , solely and nonanalytically dependent on the transitional parameter γ , can be used as a measure of how far the transition from regular to chaotic behavior has progressed [47, 51].

Furthermore of interest is the Fourier transform of the two-level correlation function, the two level form factor $b(k, \gamma)$. It can exactly be determined from R_2 and displays a robust and measurable feature of a chaotic spectrum, known as correlation hole. The transitional model provides a precise description of the correlation hole and how it will disappear when the system crosses over to a regular spectrum.

This characteristic is already observable in atomic as well as molecular systems and should be furthermore measurable in small mesoscopic systems at small temperatures.

The system properties mentioned so far concentrate on the translational invariant part of the model in the bulk of the spectrum. However, as this model breaks spontaneously the unitary symmetry associated with the translational invariance [51], the validity could be also extended to include the energy level edges. The theoretical predictions from this full model cover the appearance of long-range order oscillations in the correlation functions, which will appear as a qualitative change in long-range statistical quantities.

An important generalization of this model is the inclusion of an external parameter and its interaction with physical systems exhibiting transitional level statistics. As many systems (for example disordered conductors) are usually measured with an external field present, the importance of this extension becomes immediately obvious. This extension also allows to study new directions, as correlation functions solely depending on the external parameter can be analyzed. One fundamental result is the occurrence of a characteristic quantity in the external perturbation describing the sensitivity of the eigenvalues to the external perturbation. In case of a pure Wigner-Dyson level statistic system a well established relation exists using a fluctuation-dissipation type of equation which interprets the level sensitivity as conductance,

as originally pointed out by Thouless [96, 97]. The scaling with the characteristic quantity results in universal correlation functions. However, the generalization to transitional statistics will result in a breakdown of this universality at larger disorder (soft chaotic spectra). Furthermore, the relation of the level sensitivity to measurable quantities becomes unclear but some relation to the conductance should remain.

5.2 Relevance of the Model for Physical Systems

The questions approached by the transitional model are relevant for a large variety of physical systems. One very important example are disordered conductors. This topic has been widely investigated both experimentally and theoretically, because the disorder-induced metal-insulator transition is important to our understanding of disordered materials. Furthermore, the phenomenon of localization is fundamental in its own rights. In particular, the connection between disorder and dimensionality, which has been shown to be nontrivial in Abrahams et al. [98] by exhibiting a phase transition only for $d > 2$, has not been understood on a fundamental level.

The general belief in the community is that the metal-insulator transition at $d=3$ is sufficiently described by the 3-dimensional tight-binding Anderson Hamiltonian [99]. While the measurable physical properties like the critical exponents have been extensively investigated, a clear picture about the underlying energy level structure is missing so far. However, there have been several studies of the eigenlevel statistics of the 3D tight binding Anderson Hamiltonian [48, 100, 101] which clearly suggest that the transitional model describes the eigenvalue spectrum of the metal-insulator transition quite well. In particular, the Δ_3 -statistics in figure 3.6 exhibits the same behavior as in reference [48]. It should be noted that not only the large N linear term is present, but that also for small N the agreement is remarkable.

Reference [53] investigates the nearest neighbour spacing distribution, where the results show a very similar 'maximum shifting' behavior as observed in the present

model in figure 3.5. Also visible from figure 3.5 is the intersection of all curves in one point, which is also observed in other efforts. The appearance of a common 'scaling' point raises the intriguing possibility of a universal behavior in the critical regime and therefore an independent statistical distribution [63, 64]. This is also clearly supported by numerical investigations which indicate scale invariance at the critical point. The scaling hypothesis in [53] focusses on the decay exponent of the spacing distribution for large arguments. A similar approach has also been undertaken for the present transitional model [82]. At the moment this is still an undecided and, thus, strongly researched topic.

In summary, the agreement both in qualitative as well as in quantitative respect between the energy level statistics in the presented transitional random matrix model and the 3D tight binding Anderson model is quite good. However, there are recent indications that the spectral statistics at the transition point are universal and that they follow a unique statistical distribution [63, 53].

Another area of physics, where a transitional model is highly desirable, is quantum chaos theory. Usually, chaotic systems can only theoretically be approached in semiclassical terms based on the Gutzwiller trace formula. The Gutzwiller trace formula allows to calculate the density of states as a sum over closed orbits. This results therefore in direct information about the energy spectrum, which makes a comparison possible. Furthermore, many quantum chaos systems permit numerical treatment, so that statistical analysis is the preferred way of comparison.

Even though a direct link between standard random matrix theory and complete (hard) quantum chaos is so far not established, there is no serious doubt in the scientific community that the Wigner-Dyson statistics govern the energy levels of complete chaotic quantum systems. Since many systems also exhibit transitional spectral features, the interest in a suitable random matrix theory is natural. The hydrogen atom in a strong magnetic field represents an important and well studied

problem [102, 103], and the statistical energy level properties are very similar to the herein studied transitional model, when the strength of the applied magnetic field is increased.

Both the nearest neighbour spacing distribution and the Δ_3 -statistics have the same qualitative behavior as the transitional model. A comparison with quantum billiards, however, gives a different picture. The Δ_3 -statistics show for integrable systems a cutoff point, from which on the function remains a constant. This cutoff point can be explained by the presence of a smallest periodic orbit in the billiard. As long as the level number is below the one of the smallest periodic orbit number, the behavior is Poisson-like, but when it is crossed, the Δ_3 -statistics level off.

This picture is more involved in the case of chaotic systems, even though the indications point into the direction of a logarithmic increase of the Δ_3 -statistics and a cutoff at a specific point [104]. It is interesting to note that this behavior is also observed for the transitional model in case of the integration over the entire kernel. The cutoff appears, as soon as the origin is included in the integration limits.

The field of mesoscopic physics knows many applications, where transitional statistics are of importance. Well known examples are Aharonov-Bohm rings and quasicrystals. One important system are small metallic particles. The interest in these systems started with the investigations of Gor'kov and Eliashberg [32] who already suggested that random matrix theory applies to this type of system.

The correlation hole proposed in chapter 3 should already be an observable and robust feature. Measurements have so far been limited to atomic or molecular systems. However, this property should be also observable in systems of mesoscopic size. It is quite important to choose temperatures and particle size small enough in order to avoid continuously broadened levels, which would not exhibit level correlations. For a metallic system the assumption of an electron gas would lead to reasonably sharp levels for about 50 nanometer-particles with the temperature in the millikelvin range.

Of further interest would be the choice of metallic particles, which upon doping with another material become insulating. The correlation hole would become smaller, as predicted by equation 3.79.

5.3 Comparison with Other Approaches

The generalization of random matrix models to include transitional statistics has also been pursued with other approaches. It is quite important for the understanding of these systems to explain similarities and differences between these approaches and the presented model.

Banded random matrices have been a well studied approach for expanding standard random matrix theory. The leading idea in this model is to weight the matrix elements depending on their distance from the main diagonal of the random matrix. In case this decay is strong, only the eigenvalues on the main diagonal will remain recovering Poissonian statistics. In case that the drop-off is extremely small, the matrix elements will have approximately the same size recovering Wigner's ansatz. This transition occurs as a function of the parameter b/N , where b is the bandwidth and N is the number of eigenvalues [35, 61].

The model can be mapped onto a nonlinear σ -model and solved exactly for $N = \infty$ [59, 60]. The solution exhibits a smooth crossover from Wigner-Dyson to Poisson, and transitional features seem to be absent. The analogous behavior of the parameter ξ/L in quasi one-dimensional systems, where ξ is the localization length and L the system size, might suggest a suitable applicability for these systems.

The preferential basis model [55], already mentioned in the second chapter, is also capable of describing the statistical crossover from Wigner-Dyson to Poisson. It introduces a unitarity breaking term into the well known Wigner probability density.

$$P(H) = C_n \exp(-\text{Tr} H^2 - \hbar^2 N^2 \text{Tr}([\Lambda, H][\Lambda, H]^\dagger)) \quad (5.2)$$

The matrix Λ determines a preferential direction in matrix space and will be aver-

aged out. After averaging the model will regain unitary symmetry, even though basis preference is implicitly contained in the model. This procedure is quite reminiscent of the transitional model treated here. The construction of the theory conserves unitary symmetry as seen from the JPD in equation 1.18, but the transition from a strongly to a weakly confining potential suggests also here a spontaneously broken symmetry. Other results also underline the strong correspondence between these two models. The most striking similarity consists of the fact that in the critical regime of the preferential basis model the two point correlation functions, as given by equation 3.61, are identical. The preferential basis model shows three different regimes, which are separated by a product of the number of eigenvalues times a nontrivial parameter determined by the coefficient of the original symmetry breaking term. This is analogous to the consideration of the product γN in the weakly confining potential model, as suggested in chapter 3.

In summary, the strong similarity with the preferential basis model supports the picture of the $N = \infty$ model to characterize the critical regime and the parameter γN for describing the whole transition.

As in the transitional model, the approach by Kravtsov and Canali [51] is also based on the global maximum entropy approach and involves the solution of the mean field equation 2.19. It concentrates on the aspects of the unitary symmetry breaking and uses a similar weakly confining potential as in 3.1 to derive the correlation functions. It recovers the bulk kernel 3.61 of the transitional model and also the off-diagonal oscillations 3.101. However, the authors concentrate on the aspect of the spontaneous breaking of the unitary symmetry and associate the linear term in the number variance or Δ_3 -statistics with models which explicitly break the symmetry.

As order parameter of this transition the coefficient of the linear term was consequently suggested. The similarity in the model construction makes the model treated herein and the mean field model in most aspects identical. Therefore, the numerical

investigation of the mean field model [47] can be used without hesitation to additionally support claims in here.

5.4 Conclusion and Outlook

In summary, the transitional model with weakly confining potential has provided us with new insight in the cross-discipline area of transitional statistics. The two point correlation function has exactly been determined, an expression which has appeared since from other approaches [55], and the statistical analysis contains many features of a transition in spectral statistics visible in many physical systems.

In particular, the analysis was able to exactly predict a linear term in the number variance and the Δ_3 -statistics. This linear term has been clearly observed [48] and presents a sign of spontaneously broken symmetry. Even though it lacks so far microscopic foundation, one can seriously believe that the model contributes to unsolved questions concerning the spectral aspects of the metal-insulator transition.

The scope of the original model has been extended to include parametric correlations and it has been shown that universality present in the weakly localized or chaotic regime will break down in the appropriately scaled velocity correlator, when the disorder is strong enough [105]. The level sensitivity, partially connected to the conductance, depends on the height of a peak smeared out by disorder.

These results demonstrate two separate aspects. First of all, the weakly confining transitional model, which would already have been of interest from the mathematical physics point of view alone, shows unique and interesting phenomena compared to standard random matrices. Secondly, the actual physical model about energy level distributions is quite successful in explaining real level distribution obtained either from numerical or experimental work.

While the success of this model is encouraging, further investigations are desirable. The full two-level properties certainly warrant additional studies. Of particular inter-

est are the long-range oscillations which are the manifestations of the spontaneously broken unitary symmetry [51]. The long-range properties show a constant behavior, once the origin is included in the integration range. One can speculate that the special nature of the origin is correlated to special points in the spectrum. The band edge of a disordered solid state system would be one example, and a statistical study of numerical eigenvalues of a typical system (tight binding Anderson model) including the band edges could support this point.

Another important case would be the smallest periodic orbit of a closed quantum billiard system (chaotic or regular). These systems show a clear cutoff in their Δ_3 -statistics at a number N which is associated with the smallest periodic orbit. This behavior can be achieved from the current model by using a shifted complete two level correlation function.

An additional topic is to expand the parametric correlation calculations. The expansion of the number operator 4.44 for finite N to nonzero values of the energy, as discussed in appendix B, would lead to the straightforward investigation of the exact parametric number variance and an improved treatment of the level sensitivity. However, there is no clear way towards this goal, as the large but finite N nonzero energy asymptotics of the q -Hermite polynomials are presently not known.

Furthermore, the connection between the level sensitivity and the conductance, usually known as Thouless relation, might be extended to stronger disorder as suggested by the presented model.

A long outstanding topic is the connection between this transitional statistics model with microscopic, i.e. Hamiltonian, formulations. The encouraging resemblance between this model and the preferential basis model could eventually help to clarify this point.

APPENDIX A THE TWO-LEVEL FORM FACTOR

The calculation of the two level form factor in 3.79 is quite central to the discussion of the correlation hole and the associated fourier transform we were not able to find in standard mathematical references. Therefore a more detailed calculation has been added in this appendix.

The integral to calculate is

$$I = \int_{-\infty}^{\infty} \frac{\sin^2(ax)}{\sinh^2(bx)} e^{ikx} dx \quad (\text{A.1})$$

$$= \frac{1}{b} \int_{-\infty}^{\infty} \frac{\sin^2(ax/b)}{\sinh^2(x)} e^{i(k/b)x} dx \quad (\text{A.2})$$

The sinus function can be rewritten with the complex exponential function resulting in

$$I = -\frac{1}{4b} \int_{-\infty}^{\infty} \frac{e^{i(k+2a)y/b}}{\sinh^2(y)} dy + \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iky/b}}{\sinh^2(y)} dy - \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{i(k-2a)y/b}}{\sinh^2(y)} dy \quad (\text{A.3})$$

$$= I1 + I2 + I3 \quad (\text{A.4})$$

These integrands have now a singularity at $y=0$, in contrast to the original integrand. They have furthermore singularities at $y = i\pi n$, so that Cauchy's theorem can be applied.

Studying the integral I2 in more detail one can see that the standard half circle contour is adequate. For $k \neq 0$ the contour can be closed over the upper half plane resulting in

$$I2 = \frac{1}{2b} 2\pi i \sum_n \text{Res}_{z \rightarrow i\pi n} \left(\frac{e^{ikz/b}}{\sinh^2(z)} \right) \quad (\text{A.5})$$

Using the formula

$$\text{Res}_{z \rightarrow a} f(z) = \lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \quad (\text{A.6})$$

one gets

$$Res_{z \rightarrow i\pi n} \frac{e^{ikz/b}}{\sinh^2(z)} = \lim_{z \rightarrow i\pi n} [2(z - i\pi n) \frac{e^{ikz}}{\sinh^2(z)} (1 - (z - i\pi n) \frac{\cosh(z)}{\sinh(z)} \quad (\text{A.7})$$

$$+ (z - i\pi n)^2 \frac{ike^{ikz}}{\sinh^2(z)}] \quad (\text{A.8})$$

Expanding the hyperbolic functions around the singularity 2nd order gives then

$$Res_{z \rightarrow i\pi n} \left(\frac{e^{ikz/b}}{\sinh^2(z)} \right) = i \frac{k}{b} e^{-\pi n k/b} \quad (\text{A.9})$$

$$i \frac{k}{b} \left(\frac{1}{1 - e^{k\pi/b}} - \frac{1}{2} \right) \quad (\text{A.10})$$

and (taking the principal value)

$$I2(k > 0) = \frac{\pi k}{b^2} \left(\frac{1}{2} - \frac{1}{1 - e^{-|k|\pi/b}} \right) \quad (\text{A.11})$$

For $k < 0$ the singularities are at $y = -i\pi n$ and the contour closes over the lower half plane. The above argument leads then to the same residue and the integral is

$$I2 = \frac{\pi |k|}{b^2} \left(\frac{1}{2} - \frac{1}{1 - e^{-|k|\pi/b}} \right) \quad (\text{A.12})$$

The integrals I1 and I3 are quite analogous to I2 with the results

$$I1 = \frac{\pi}{2b^2} |2a + k| \left(\frac{1}{1 - e^{-|k+2a|\pi/b}} - \frac{1}{2} \right) \quad (\text{A.13})$$

and

$$I3 = \frac{\pi}{2b^2} |k - 2a| \left(\frac{1}{1 - e^{-|k-2a|\pi/b}} - \frac{1}{2} \right) \quad (\text{A.14})$$

This results in

$$I = \frac{\pi}{2b^2} |k + 2a| \left(\frac{1}{1 - e^{-|k+2a|\pi/b}} - \frac{1}{2} \right) \quad (\text{A.15})$$

$$+ \frac{\pi |k|}{b^2} \left(\frac{1}{2} - \frac{1}{1 - e^{-|k|\pi/b}} \right) \quad (\text{A.16})$$

$$+ \frac{\pi}{2b^2} |k - 2a| \left(\frac{1}{1 - e^{-|k-2a|\pi/b}} - \frac{1}{2} \right) \quad (\text{A.17})$$

$$(\text{A.18})$$

from which we finally obtains

$$b(k) = \frac{1}{4\pi}[(|k| + 2\pi)\coth[(|k| + 2\pi)\pi/\beta] \quad (\text{A.19})$$

$$+ (|k| - 2\pi)\coth[(|k| - 2\pi)\pi/\beta] \quad (\text{A.20})$$

$$- 2|k|\coth(|k|\pi/\beta)] \quad (\text{A.21})$$

This is the expression for the two level form factor.

APPENDIX B THE VELOCITY CORRELATOR

To calculate the velocity correlator from the standard form of the density-density correlation function a rather technical procedure was followed, which is the reason it appears in this appendix.

The correlator is defined as

$$C(E, E', X, X') = \int^E \int^{E'} \frac{\partial^2}{\partial X \partial X'} S(E, E', X, X') \quad (\text{B.1})$$

where

$$S(E, E', X, X') = c w(\sqrt{c}E) w(\sqrt{c}E') \sum_{m=0}^{N-1} \sum_{n=N}^{\infty} h_n(\sinh(\sqrt{c}E)) h_n(\sinh(\sqrt{c}E')) h_m(\sinh(\sqrt{c}E)) h_m(\sinh(\sqrt{c}E')) e^{(m_q - n_q)\omega(X - X')^2} \quad (\text{B.2})$$

The constant c and ω are scaled in a way so that $S(E, E', 0, 0) = \sigma(E) \equiv 1$. From equation () they are $c = \gamma\pi^2/2N_q$ and $\omega = c/\mu$. The calculation of $C(E, E', X, X')$ has to be limited to $E = E' = 0$, because for finite E the q -Hermite polynomials are very difficult to estimate. The reason is that the polynomials h_n exhibit different behaviour depending on whether the argument $\sinh(\sqrt{c}E)$ is larger or smaller than γn . In case it is smaller the curve is very close to the Hermite polynomials, compared to a large q -deformation for larger arguments. An approximation procedure capturing this crossover in the polynomials has not been found so far, but it would be certainly of interest in order to expand the results obtained here.

For $E = E' = 0$ the velocity correlator has the following form

$$C(X, X') = c \sum_{m=0}^{N-1} \sum_{n=N}^{\infty} \left(\int_{-\infty}^0 h_n(\sinh(\sqrt{c}E)) h_m(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \right) \left(\int_{-\infty}^0 h_n(\sinh(\sqrt{c}E')) h_m(\sinh(\sqrt{c}E')) w(\sqrt{c}E') dE' \right) \frac{\partial^2}{\partial X \partial X'} e^{(m_q - n_q)\omega(X - X')^2} \quad (\text{B.3})$$

The derivatives in X are fairly simple to calculate, so the problem reduces to evaluation of

$$\int_{-\infty}^0 h_n(\sinh(\sqrt{c}E)) h_m(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \quad (\text{B.4})$$

For $q = 1$ these integrals can be approximated by using the large N expansion for the normalized Hermite polynomials H_n

$$H_{2n}(\sqrt{2n}E) = (-1)^n \frac{(2c)^{1/4}}{\sqrt{\pi}} \frac{1}{n^{1/4}} \cos(\sqrt{2c(4n+1)}E) e^{cE^2} \quad (\text{B.5})$$

$$H_{2n+1}(\sqrt{2n}E) = (-1)^n \frac{(2c)^{1/4}}{\sqrt{\pi}} \frac{1}{n^{1/4}} \sin(\sqrt{2c(4n+3)}E) e^{cE^2} \quad (\text{B.6})$$

Note that the factor $\sqrt{c}$ is replaced here by $\sqrt{2c}$, which is necessary because of relation 3.24. This results in the integral representation of the density-density correlation function [88] ($c \simeq \frac{1}{N}$)

$$S(E, E', X - X') = \int_0^1 ds \int_1^\infty dt \cos(\pi(E - E')s) \cos(\pi(E - E')t) e^{(s^2 - t^2)(X - X')^2 \pi^2 / 2\mu} \quad (\text{B.7})$$

The velocity correlator is then

$$C(E, E', X - X') = \int_0^1 ds \int_1^\infty dt J_{s,t} (2 \frac{\pi^2}{2\mu} (t^2 - s^2) - 4 (\frac{\pi^2}{2\mu})^2 \frac{X^2}{\mu} (t^2 - s^2)^2) e^{(s^2 - t^2) X^2 / 2\mu} \quad (\text{B.8})$$

where the expression $J_{s,t}$ stands for

$$J_{s,t} = \int_{-\infty}^E \int_{-\infty}^{E'} \cos(\pi(E - E')s) \cos(\pi(E - E')t) dE dE' \quad (\text{B.9})$$

The term $J_{s,t}$ can be evaluated by integration and with some straightforward trigonometric manipulations to be

$$J_{s,t} = \frac{2}{\pi^2} \left[\frac{\cos(\pi(E - E')(s - t))}{(s - t)^2} + \frac{\cos(\pi(E - E')(s + t))}{(s + t)^2} \right] \quad (\text{B.10})$$

As expected, this depends only on the difference $E - E'$. This expression is needed to recover equation 4.75 in the text.

The integral over s and t is dominated by the region $s \approx t \approx 1$. Hence one can transform the integral with the substitutions $s^2 - t^2 = k$; $s^2 + t^2 = m$ approximately

into the expression (the integral over $s^2 + t^2$ is roughly estimated with $s^2 + t^2 \approx 2$)

$$\propto \int_0^1 \frac{(1 - \cos(\pi k(E - E')))}{k} e^{2k(X - X')^2 \pi^2 / 2\mu} dk \quad (\text{B.11})$$

By noting that the extension of the integration region to negative numbers (note that always $t > s$) will result in the absolute value for k , and that the extension to $k \rightarrow \pm\infty$ is only appropriate for $X^2 \gg \frac{2\mu}{\pi^2}$ the integral results in

$$\int_{-\infty}^{\infty} dk \frac{1 - \cos(k(E - E'))}{|k|} e^{-|k|(X - X')^2 \pi^2 / 2\mu} \quad (\text{B.12})$$

Apart from the constants, this is the expression for $q=1$ needed for the calculation of the parametric number variance in 4.44. Hence this argument can with appropriate restrictions on X , as seen above, recover the hydrodynamic approximation. It is also easily possible to recover the well known expression 4.29 from here.

We are mostly interested in $E = E' = 0$, because it gives an X -dependent quantity, without energy complications. Then equation B.9 transforms into

$$J_{s,t} = \frac{4}{\pi^2} \frac{s^2 + t^2}{(s^2 - t^2)^2} \quad (\text{B.13})$$

and therefore

$$C(X - X') = \frac{1}{\mu} \left[\int_0^1 ds \int_1^\infty dt \frac{t^2 + s^2}{t^2 - s^2} e^{(s^2 - t^2)(X - X')^2 \pi^2 / 2\mu} \right] \quad (\text{B.14})$$

$$- \frac{(X - X')^2 \pi^2}{2\mu} \int_0^1 ds \int_1^\infty dt (t^2 + s^2) e^{(s^2 - t^2)(X - X')^2 \pi^2 / 2\mu} \quad (\text{B.15})$$

The conjunction is now that the same expression is valid for $q < 1$, the only difference being the upper integration limit. However, this has to be shown explicitly.

The integral 4.38 can exactly be evaluated for $E = E' = 0$. First the evaluation will concentrate on the case $q=1$, which leads to the result 4.45 in the text. This does not only show the procedure, but it also leads to the relation which is a result previously not obtained. As before, we will have to evaluate

$$\int_{-\infty}^0 H_n(\sqrt{2c}E) H_m(\sqrt{2c}E) e^{-2cE^2} dE \quad (\text{B.16})$$

Except for the different integration limit this expression represents the orthogonality relation for Hermite polynomials. Then we see immediately that in case of both n and m even the integral vanishes because of the symmetric integrand.

$$\int_{-\infty}^0 H_{2n} H_{2m} e^{-cE^2} dE = \frac{1}{2} \int_{-\infty}^{\infty} H_{2n} H_{2m} e^{-cE^2} dE = \delta_{nm} \quad (B.17)$$

Recalling the well known 'raising' and 'lowering' operators for Hermite polynomials we have

$$\begin{aligned} \partial_E H_n(\sqrt{2c}E) &= 2\sqrt{2cn} H_{n-1}(\sqrt{2c}E) \\ e^{cE^2} \partial_E \{H_n(\sqrt{2c}E) e^{-cE^2}\} &= -\sqrt{2c} H_{n+1}(\sqrt{2c}E) \end{aligned} \quad (B.18)$$

Taking the case that n is even and m odd first, the use of B.18 and partial integration gives

$$\int_{-\infty}^0 \int_{-\infty}^0 H_{2n+1}(\sqrt{2c}E) H_{2m}(\sqrt{2c}E) e^{-cE^2} dE \quad (B.19)$$

$$= 2(2n+1) \int_{-\infty}^0 \int_{-\infty}^0 H_{2n}(\sqrt{2c}E) H_{2m-1}(\sqrt{2c}E) e^{-cE^2} dE \quad (B.20)$$

$$= 2^2(2n+1)(2m-1) \int_{-\infty}^0 \int_{-\infty}^0 H_{2n-1}(\sqrt{2c}E) H_{2m-2}(\sqrt{2c}E) e^{-cE^2} dE \quad (B.21)$$

$$= \dots = \quad (B.22)$$

$$= 2^{2m}(2n+1)(2n-1)\dots(2n-2m+1)(2m-1)(2m-3)\dots 1 \quad (B.23)$$

$$\int_{-\infty}^0 H_0 H_{2n-2m+1}(\sqrt{2c}E) e^{-cE^2} dE \quad (B.24)$$

With

$$\int_{-\infty}^0 H_{2n-2m+1}(\sqrt{2c}E) e^{-cE^2} dE = H_{2n-2m-2}(0) \quad (B.25)$$

and

$$H_{2n}(0) = (-1)^n 2^n (2n-1)!! \quad (B.26)$$

we get

$$\int_{-\infty}^0 H_{2n}(\sqrt{2c}E) H_{2m}(\sqrt{2c}E) e^{-cE^2} dE = \frac{2^{n+m} (2n+1)!! (2m-1)!!}{2(2n-2m-1)} \quad (B.27)$$

Taking the normalization into account

$$\begin{aligned} & \frac{1}{\pi 2^{2n+2m+1} (2n+1)! (2m)!} \left[\int_{-\infty}^0 H_{2n+1}(\sqrt{2c}E) H_{2m}(\sqrt{2c}E) e^{-cE^2} dE \right]^2 \quad (\text{B.28}) \\ &= \frac{1}{2\pi} \frac{[(2n+1)!!]^2 [(2m-1)!!]^2}{(2n+1)! (2m)! (2n-2m-1)^2} \quad (\text{B.29}) \end{aligned}$$

The case m even and n odd works out in an analogous fashion (note that $n > m+2$)

$$\begin{aligned} & \frac{1}{\pi 2^{2n+2m+1} (2n+1)! (2m)!} \left[\int_{-\infty}^0 H_{2n}(\sqrt{2c}E) H_{2m+1}(\sqrt{2c}E) e^{-cE^2} dE \right]^2 \quad (\text{B.30}) \\ &= \frac{1}{2\pi} \frac{[(2n)!!]^2 [(2m+1)!!]^2}{(2n)! (2m+1)! (2n-2m-3)^2} \quad (\text{B.31}) \end{aligned}$$

This term can be expressed with Γ functions as

$$\frac{[(2n+1)!!]^2 [(2m-1)!!]^2}{(2n+1)! (2m)! (2n-2m-1)^2} = \frac{1}{\pi} \frac{\Gamma(n+3/2) \Gamma(m+1)}{\Gamma(n+1) \Gamma(m+3/2)} \quad (\text{B.32})$$

and

$$\frac{[(2n)!!]^2 [(2m+1)!!]^2}{(2n)! (2m+1)! (2n-2m-3)^2} = \frac{1}{\pi} \frac{\Gamma(m+3/2) \Gamma(n+1)}{\Gamma(m+1) \Gamma(n+3/2)} \quad (\text{B.33})$$

The velocity correlator is then

$$C(X-X') = \frac{1}{2\pi^2} \sum_{m=0}^{N/2} \sum_{n=N/2}^{\infty} \frac{\Gamma(n+3/2) \Gamma(m+1)}{\Gamma(n+1) \Gamma(m+3/2)} \frac{1}{(2n-2m-1)^2} \quad (\text{B.34})$$

$$\frac{\partial^2}{\partial X \partial X'} e^{(m-n)\omega(X-X')^2} \quad (\text{B.35})$$

$$+ \frac{1}{2\pi^2} \sum_{m=1}^{N/2-1} \sum_{n=N/2+1}^{\infty} \frac{\Gamma(m+3/2) \Gamma(n+1)}{\Gamma(m+1) \Gamma(n+3/2)} \frac{1}{(2n-2m-3)^2} \quad (\text{B.36})$$

$$\frac{\partial^2}{\partial X \partial X'} e^{(m-n)\omega(X-X')^2} \quad (\text{B.37})$$

Using an integral instead of the sum and estimating $\frac{\Gamma(m+3/2) \Gamma(n+1)}{\Gamma(m+1) \Gamma(n+3/2)} = \frac{m+\pi/4}{n+\pi/4}$ (the approximation $\sqrt{\frac{m}{n}}$ leads to the same) one obtains

$$C(X-X') = \frac{1}{2\pi^2 N_q} \int_{m=0}^{N/2} dm \int_{n=N/2}^{\infty} dn \frac{\sqrt{m+\pi/4}}{\sqrt{n+\pi/4}} \frac{1}{(2n-2m-1)^2} \quad (\text{B.38})$$

$$\frac{\partial^2}{\partial X \partial X'} e^{((2m)-(2n+1))\omega(X-X')^2} \quad (\text{B.39})$$

$$+ \frac{1}{2\pi^2 N_q} \int_{m=1}^{N/2-1} dm \int_{n=N/2+1}^{\infty} dn \frac{\sqrt{n + \pi/4}}{\sqrt{m + \pi/4}} \frac{1}{(2n - 2m - 3)^2} \quad (\text{B.40})$$

$$\frac{\partial^2}{\partial_X \partial_{X'}} e^{((2m+1)-2n)\omega(X-X')^2} \quad (\text{B.41})$$

Using $s^2 = 2Nm$, $t^2 = 2Nn$, $c = \frac{\pi^2}{2N}$, $\omega = \frac{\varepsilon}{\mu}$ one gets (Neglecting terms of order $1/N$)

$$C(X - X') = \frac{2}{\pi^2} \int_0^1 \int_1^{\infty} \frac{s^2 + t^2}{(t^2 - s^2)^2} \frac{\partial^2}{\partial_X \partial_{X'}} e^{(s^2 - t^2)\pi^2(X - X')^2/\mu} \quad (\text{B.42})$$

This represents the integral approximation for the velocity correlator for $\gamma = 0$.

Now we can switch to $q < 1$. Similar to the previous discussion the integrals with both n and m even vanish

$$\int_{-\infty}^0 h_{2n} h_{2m} e^{-cE^2} dE = \frac{1}{2} \int_{-\infty}^{\infty} h_{2n} h_{2m} e^{-cE^2} dE = {}^{n>m} 0 \quad (\text{B.43})$$

The 'raising' and 'lowering' operations are [91]

$$\begin{aligned} D_q h_n(\sinh(\sqrt{c}E)) &= 2\sqrt{c} q^{-(n-1)/2} \frac{1-q^n}{1-q} h_{n-1}(\sinh(\sqrt{c}E)) \\ h_{n+1}(\sinh(\sqrt{c}E)) &= -\frac{1-q}{q^{n/2-1}\sqrt{c}} D_q \{h_n(\sinh(\sqrt{c}E))w(\sqrt{c}E)\} \end{aligned} \quad (\text{B.44})$$

The partial integration rule can be expanded to include the q -Hermite polynomials [76] resulting in

$$\begin{aligned} & \int_0^{\infty} h_{2n+1}(\sinh(\sqrt{c}E)) h_{2m}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \\ &= -\frac{2(1-q)}{q^{m+1}} q^{-n+1/2} \frac{1-q^{2n+1}}{1-q} \int_{-\infty}^0 h_{2m-1}(\sinh(\sqrt{c}E)) h_{2n}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \\ &= \frac{2^2(1-q)^2}{q^{2m+2n+2}} (2n+1)_q (2m-1)_q \int_{-\infty}^0 h_{2m-2}(\sinh(\sqrt{c}E)) h_{2n-1}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \\ &= \dots \\ &= \frac{2^{2m}(1-q)^{2m}}{q^{2m} q^n (2m+1) q^{n(2n+1)}} (2n+1)_q (2n-1)_q \dots (2n-2m+1)_q (2m-1)_q (2m-3)_q \dots 1 \\ & \int_{-\infty}^0 h_0(\sinh(\sqrt{c}E)) h_{2n-2m+1}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \end{aligned} \quad (\text{B.45})$$

Now with

$$\int_{-\infty}^0 h_0(\sinh(\sqrt{c}E)) h_{2n-2m+1}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \quad (\text{B.46})$$

$$= -\frac{1-q}{q^{2n-2m+1}\sqrt{c}} h_{2n-2m+1}(0) w(0) \quad (\text{B.47})$$

and

$$h_{2n}(\sqrt{c}E \rightarrow 0) = \frac{\prod_{i=1}^{2n-2m+1} (1 - q^{2i-1})}{q^{(2n-2m+1)^2/2}} \quad (\text{B.48})$$

one arrives at

$$\int_0^\infty h_{2n+1}(\sinh(\sqrt{c}E)) h_{2m}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \quad (\text{B.49})$$

$$= \frac{(1-q)^{(2n+2m+1)/2}}{q^{2m+m(2m+1)+(2n+1)(n+1)}} \frac{((2n+1)_q!!)((2m-1)_q!!)}{(2n-2m-1)_q} \quad (\text{B.50})$$

where $(2n+1)_q!! = \frac{(1-q^{2n+1})(1-q^{2n-1})\dots(1-q)}{(1-q)^n}$ (for simplicity $(-1)_q! = 1$) was used. Also the natural extension of the factorial $n_q! = \frac{(1-q^n)(1-q^{n-1})\dots(1-q)}{(1-q)^n}$ will be of help. Including the normalization

$$\left(\frac{q^{(2n+1)(2n+2)/2+m(2m+1)}}{\gamma(q; q)_\infty (q; q)_{2n+1} (q; q)_{2m}} \right)^{1/2} \quad (\text{B.51})$$

and the relation

$$(2n-2m-1)_q = \frac{1}{q^{2m}} ((2n-1)_q - (2m)_q) \quad (\text{B.52})$$

the integral becomes

$$\left[\int_{-\infty}^0 h_{2n+1}(\sinh(\sqrt{c}E)) h_{2m}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \right]^2 \quad (\text{B.53})$$

$$= \frac{w^2(0)}{\gamma^2(q; q)_\infty} \frac{q^{2n+2m} ((2n+1)_q!!)^2 ((2m-1)_q!!)^2}{(2n+1)_q! (2m)_q! ((2n-1)_q - (2m)_q)^2} \quad (\text{B.54})$$

The case m odd and n even ($n > m+2$) is treated analogously with the result

$$\left[\int_{-\infty}^0 h_{2m+1}(\sinh(\sqrt{c}E)) h_{2n}(\sinh(\sqrt{c}E)) w(\sqrt{c}E) dE \right]^2 \quad (\text{B.55})$$

$$= \frac{w^2(0)}{\gamma^2(q; q)_\infty} \frac{q^{2n+2m} ((2n-1)_q!!)^2 ((2m+1)_q!!)^2}{(2n)_q! (2m+1)_q! ((2m-3)_q - (2m)_q)^2} \quad (\text{B.56})$$

The final expression for $C(X, X')$ is then

$$\begin{aligned} C(X, X') &= \frac{Ac}{2\gamma} \left[\sum_{m=0}^{[N/2]} \sum_{n=[N/2]+1}^{\infty} \frac{q^{2n+2m} ((2n+1)_q!!)^2 ((2m-1)_q!!)^2}{(2n+1)_q! (2m)_q! ((2n-1)_q - (2m)_q)^2} f_{2n+1, 2m}(X, X') \right. \\ &\quad \left. + \sum_{m=1}^{[N/2]-1} \sum_{n=[N/2]+1}^{\infty} \frac{q^{2n+2m} ((2n-1)_q!!)^2 ((2m+1)_q!!)^2}{(2n)_q! (2m+1)_q! ((2n-3)_q - (2m)_q)^2} f_{2n, 2m+1}(X, X') \right] \end{aligned}$$

for $N > 2$. The function f is given by $f_{n,m}(X, X') = \frac{\beta^2}{\partial X \partial X'} e^{((2m)_q - (2n-3)_q)\omega(X-X')^2}$.

Analogously to the undeformed case it is desirable to find an integral approximation to this sum for large N .

Realizing that the expression $a_{n,m} = \frac{((2n+1)_q!)^2}{(2n)!_q!}$ has the two distinct limits $a_{n,m} \xrightarrow{\gamma \rightarrow 0} \frac{((2n+1)!)^2}{(2n)!} \approx \frac{\sqrt{n}}{\sqrt{\pi}}$ on the one hand and $a_{n,m} \xrightarrow{\gamma \rightarrow 0} \frac{1}{\sqrt{\pi\gamma}}$ on the other hand, the following approximation is natural

$$a_{n,m} = \frac{1}{\sqrt{\pi}\sqrt{n}} \quad (\text{B.57})$$

Then we can write for the sum the following approximate integral expression (Neglecting terms on the order of $1/N$)

$$\begin{aligned} C(X, X') &= \frac{1}{2\pi^2} \int_0^{N/2} dm \int_{N/2}^\infty dn \sqrt{\frac{n_q}{m_q}} \frac{q^{2n+2m}}{(2n_q - 2m_q)^2} f_{n,m}(X, X') \\ &+ \frac{1}{2\pi^2} \int_0^{N/2} dm \int_{N/2}^\infty dn \sqrt{\frac{m_q}{n_q}} \frac{q^{2n+2m}}{(2n_q - 2m_q)^2} f_{n,m}(X, X') \quad (\text{B.58}) \end{aligned}$$

Again the substitution $n_q = 2Nt, m_q = 2Ns$ and use of the relations $c = \pi^2\gamma/2N_q, \omega = \pi^2/2\mu N_q$ results then in

$$\begin{aligned} C(X, X') &= \frac{1}{\mu} \left[\int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt \frac{s^2 + t^2}{s^2 - t^2} e^{(s^2 - t^2)(X - X')\pi^2/2\mu} \right. \\ &\quad \left. - \frac{\pi^2(X - X')^2}{2\mu} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt (t^2 + s^2) e^{(s^2 - t^2)(X - X')\pi^2/2\mu} \right] \quad (\text{B.59}) \end{aligned}$$

This represents the integral approximation for the velocity correlator for $\gamma > 0$.

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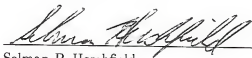
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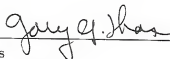
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